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Multigranulation Rough Set Methods and Applications Based on Neighborhood Dominance Relation in Intuitionistic Fuzzy Datasets

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Abstract With the redundancy and complexity of information and data, how to acquire the samples that meet the requirements is an inevitable task in data analysis. There is a general consensus that the neighborhood rough set (NRS) has become the mainstream method for data mining and knowledge classification. Whereas, the limitations still exist in the neighborhood relation for it cannot more accurately reflect the dominance relations that commonly exist in actual data, nor can it select the required data according to different conditions. Enlightened by this idea, this paper focuses on the intuitionistic fuzzy neighborhood dominance relation, which both refines the relationship between samples in the neighborhood and mines the needed samples in data analysis. On this basis, we define the neighborhood dominance rough set (NDRS) model in intuitionistic fuzzy ordered information system (IFOIS). Moreover, we establish the multigranulation neighborhood dominance rough set (MNDRS) from multiple perspectives, and discuss related properties between NDRS and MNDRS. Meanwhile, we compare the NDRS with other rough set models from the roughness and the dependence degree viewpoints. Finally, we adopt nine UCI data sets and implement a series of experiments to illustrate the feasibility and effectiveness of the proposed models.

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1 Introduction

Rough set theory (RST), proposed by Pawlak [30] in 1982, is an effective tool for knowledge discovery and rule extraction in information systems. As an extension of classical set theory, the idea of RST is to approximate some inaccurate and uncertain concepts with known knowledge. In recent years, the computing method in RST has aroused wide attention, and its related research has been applied in many important fields, such as data mining [9, 22, 25, 38], decision analysis [20, 24, 37, 41], and medical diagnosis [23, 39, 40, 50].

In 1986, Atanassov [1] proposed intuitionistic fuzzy set (IFS), an extension of fuzzy set theory [20, 31], to accurately characterize the uncertainty of objects. In the intuitionistic fuzzy set theory, an object can be characterized not only by the membership degree, but also by the non-membership degree and hesitation degree. The introduction of IFS greatly improves the description of the object's features and makes the description of the object more accurate and specific. The combination of intuitionistic fuzzy set theory and RST produces a new rough set model to deal with intuitionistic fuzzy data sets. For example, Lu and Lei [17] designed an attribute reduction algorithm based on intuitionistic fuzzy rough set (IFRS). Huang et al. [14] extended IFRS model to two types of multi-granulation intuitionistic fuzzy rough set models, and discussed their basic properties. De et al. [27] investigated three-way decisions with intuitionistic fuzzy decision-theoretic rough sets based on point operators and many other relevant generalizations. This article is based on the intuitionistic fuzzy dataset as the research goal.

With the diversification and complexity of information and data. In order to handle these variable and complex datas flexibly, some researchers have proposed binary relations [8, 11, 19, 29] on the basis of equivalence relations, such as neighborhood relations [4, 36, 49], tolerance relations [32, 35], etc. Among these relations, the neighborhood rough set (NRS) based on neighborhood relations is very significant in handling numerical datasets, which can reduce the number of noise samples. As shown in Fig. 1b neighborhood relation, samples with the object x_i are partitioned into two groups S_1 and S_2, where the group S_2 are the noise samples whose distance from the object x_i is greater than δ. During the past years, some scholars have managed related achievements with NRS as the tool of their research. For example, Wang and Qian proposed local NRS model for attribute reduction [28]. Hu and Yu studied heterogeneous feature subset selection based on NRS [16]. Sun et al. presented feature selection using fuzzy neighborhood entropy-based uncertainty measures for fuzzy neighborhood multigranulation rough sets [10].

For an intuitionistic fuzzy ordered information system (IFOIS), the order relationship between samples is well described by dominance relations. Since the classical RST fail to process the partial order relationship of attribute domains. Therefore, Greco et al. presented the dominance-based rough set approach (DRSA) [12], and its relation takes into account the ordering properties of criteria [13]. As shown in Fig. 1a Dominance relation, the samples in the data can be roughly divided into four groups S_1, S_1, S_3 and S_4 according to the relationship between the other samples and a certain sample point x_i. The group S_1 represents the samples that are dominating x_i, and S_2 shows the samples that are dominated by x_i. Otherwise, S_3 and S_4 denote the samples that are neither dominating nor dominated by x_i. How to use the DRSA model to mine the knowledge in the order information system is a hot topic in research. Xu et al. considered attribute reduction in interval-valued fuzzy ordered decision tables [43]. Yang et al. considered dominance-based rough set approach and knowledge reductions in incomplete ordered information system [5]. And Yang and Yu et al. studied dominance-based rough set approach to incomplete interval-valued information system [47]. On the basis of single granulation dominance rough set approach (DRSA), Yang et al. introduced the models of dominance–based multigranulation rough sets [46], and many other valuable achievements.

From the rough set models mentioned above, the NRS realizes the processing of continuous data and the classification of similar samples, avoiding the influence of noisy samples and the loss of potential information. In an ordered dataset, the classification of samples is inseparable from the investigation of the partial order relationship of the data. Hence, it is necessary to refine the relationship between samples in the neighborhood to obtain the required objects more accurately. In today's life, there exists many problems such as college examinations, talent selections and housing screening, etc., which involve sorting of samples and the classification of objects. In view of the tolerance of NRS and the robustness of DRSA, how to utilize these two models to solve the issue of sample selection in practical problems is a motivation of this study.

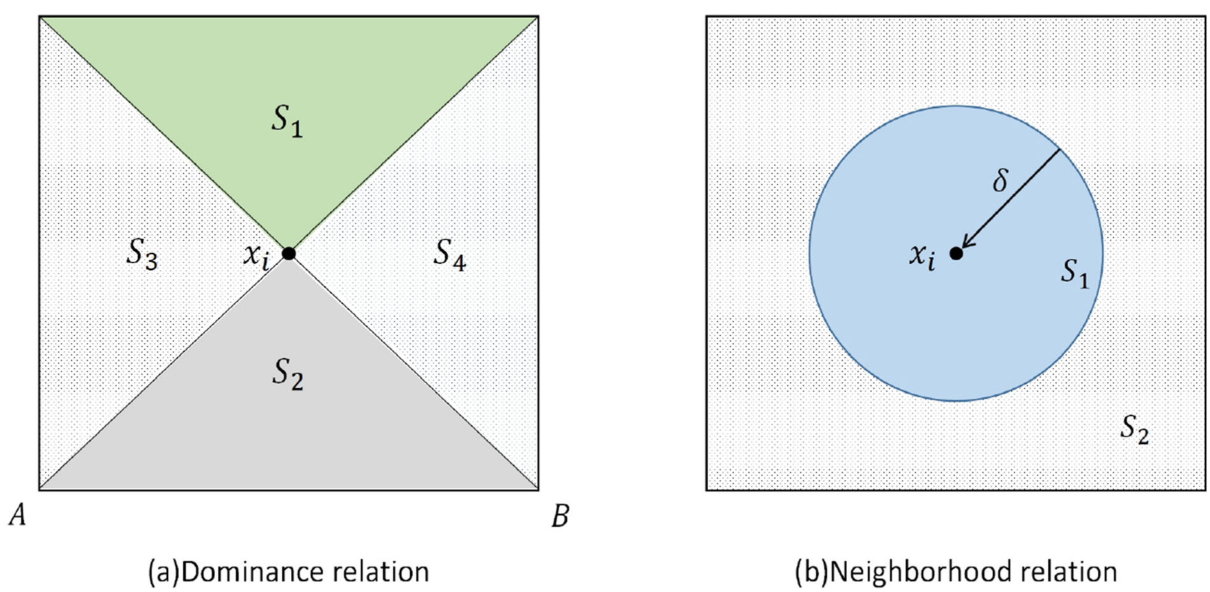


Fig. 1 The relationship among objects

Combined the idea of NRS and DRSA, the neighborhood dominance rough set (NDRS) model can be obtained. In the dataset, the neighborhood dominance relationship considers both the similarity and the partial order relation between samples, which helps to further extract samples with the same characteristics in the information granules. Through the neighborhood dominance relation, we can obtain samples with more similar features based on the target set. As shown in Fig. 2c neighborhood dominance relation, the groups S_1 and S_2 show that not only the set of objects is dominating x_i or dominated by x_i, but also the distance from x_i is less than or equal to δ. Groups S_3, S_4 represent the distance between objects and x_i is less than or equal to δ, they are neither dominated nor dominating x_i. Similarly, we can obtain the groups S_5, S_6, S_7 and S_8. They are sets of objects outside the neighborhood threshold δ which keep the partial order relationship unchanged. By altering δ, the dominance and dominated objects will change accordingly in applications. In addition, for a clearer display and an intuitive comparison between the NDRS model and other previous models, we show the details in Table 1. The distance-based rough set model proposed by Huang et al. [18] defines the distance between intuitionistic fuzzy numbers, but it fail to characterize the order relationship between samples and therefore cannot filter the qualified samples. With respect to IFIS, Bing et al. [2, 3] both considered the dominance relation in IFS and interval-valued IFS. However, the dominance relation alone cannot cope with the impact of noisy samples or the more precise requirements. On the basis of dominance relation, Zhang et al. [51] proposed generalized dominance

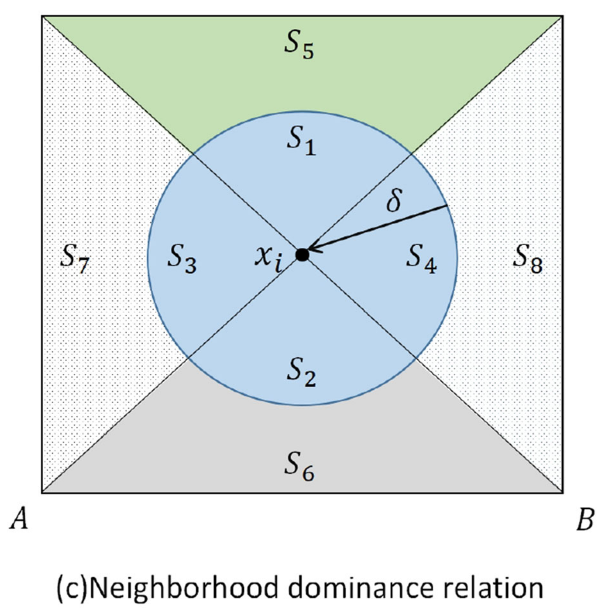


Fig. 2 The relationship among objects

relation to reduce the loss of valid information. Furthermore, Huang et al. [21] and Xue et al. [45] discussed the combination of MGRS an IFS in their research, which further expanded the application scope of IFRS. And the notion of neighborhood was introduced in the study of Xue et al. Nevertheless, the lack of research on dominance relations renders the model incapable of selecting the required samples.

The NDRS is a new hybrid model to process hybrid dataset in information systems while finding common qualified samples in the dataset. In contrast, the existing dominance-based neighborhood rough set (DNRS) is a model to handle information systems while seeking difference in the dataset, and some achievements based on DNRS model have been made to the field of attribute reduction research. In 2015, Chen et al. first put forward an important tool named dominance-based NRS and designed its attribute reduction [6]. In 2016, Chen et al. discussed a parallel attribute reduction in dominance-based neighborhood rough set model [7]. Then Sang proposed incremental approaches for heterogeneous feature selection in dynamic ordered data [8]. For one side, aiming to deeply mine the required data and handle the problem of sample selection in data analysis. For the other side, simulating the firmness and hesitation of people when making decisions. Accordingly, what we investigate is the NDRS model in IFOIS.

In practical applications, it is commonly necessary to characterize data from multiple levels and multiple aspects. However, the classical rough set model is based on a single granulation and a single level, and cannot analyze problems or mine knowledge from a multi-faceted and deep layer. Since Qian et al. [26] proposed multi-granulation rough set (MGRS), which enables the target set to be depicted by information particles in multiple granular spaces. In a consequence, the problem can be analyzed from different angles and multiple levels, so as to obtain a more satisfactory and reasonable solution. Subsequently, many scholars are devoted to the research of multigranulation rough sets [15, 29, 33]. Xu et al. proposed a generalized multigranulation rough set approach [34], and it is a generalized model between optimistic and pessimistic multigranulation rough set models. After two years, Xu et al. proposed two kinds of generalized multigranulation rough set [42] and local multigranulation decision-theoretic rough set in ordered information systems [44], in which the lower and upper approximation operators were defined and the related properties were discussed. Considering the increase or decrease of granulations, Yang et al. presented updating multigranulation rough approximations with increasing of granular structures [48]. In addition, Huang et al. extended the multigranulation theory to the IFRS [21]. Consistent with these scholars, in order to make the NDRS widely applied to intuitionistic fuzzy datasets, we

Table 1 The review and comparison of the proposed methods in IFIS

Year	Authors	Researches	Methods
2011	Huang et al.	Distance-based rough set model in IFIS and its application [20]	Dis-RSA
2012	Bing et al.	A dominance interval-valued IFRS model and its application [2]	Iv-DRSA
2013	Bing et al.	Dominance-based rough set model in IFIS [3]	DRSA
2014	Zhang et al.	Generalized dominance-based rough set model for IFIS [44]	GDRSA
2014	Huang et al.	Intuitionistic fuzzy multigranulation rough sets [17]	MGRS
2017	Xue et al.	Model of multi-granulation neighborhood rough IFS [39]	MNRS
2022	Zhang et al.	MGRS methods and applications based on NDR in IF datasets	NDRS

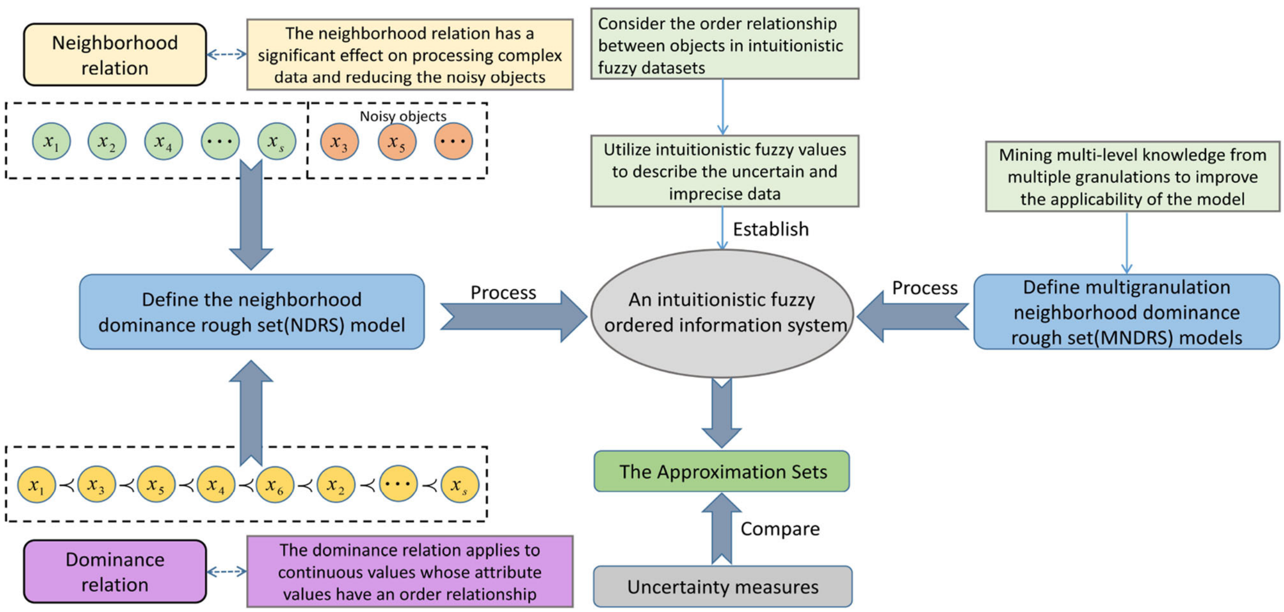


Fig. 3 System diagram of our work

extend the NDRS model from the perspective of MGRS. This extension has significant implications in mining deeper hidden knowledge and dealing with intuitionistic fuzzy datasets.

With the complexity of the data form, in view of the advantages of IFS in describing the data, we construct the NDRS model and MNDRS model to solve the problem of sample selection from single-level and multi-level perspective, and further explore the uncertainty and imprecise knowledge in data. As shown in Fig. 3, the contributions of our work are as follows: (1) A novel relation named neighborhood dominance relation both owes the classification effect of neighborhood relations and the sorting feature of dominance relations is presented. On this basis, we propose the NDRS model in IFOIS. (2) We dissect the shortcomings of the application of the NRS model, and effectively illustrate the feasibility and validity of the NDRS model according to the mathematical analysis and practical problems. (3) We extend NDRS to two types of MNDRS, and carefully discuss the connection of NDRS and MNDRS. The corresponding algorithms for computing uncertainty measures of NDRS and MNDRS are also researched. (4) Moreover, through the analysis and comparison of the experimental results, the effectiveness of NDRS and MNDRS in IFOIS is verified.

This paper is organized as follows. In Sect. 2, we review some basic concepts about NRS in IFOIS. Section 3 defines the NDRS model and its uncertainty measures in IFOIS, meanwhile, an actual case is illustrated to verify the superiority of NDRS. In addition, Sect. 4 introduces two types of MNDRS in IFOIS and related properties. And Sect. 5 mainly explores the uncertainty measures of MNDRS. In Sect. 6, we exhibit two algorithms for computing the roughness and the dependence degree of NRS, NDRS and MNDRS, respectively. Furthermore, some UCI data sets are used to verify the effectiveness of proposed theorems in Sect. 7. Eventually, Sect. 8 ends up with the summarization of the paper and proposal of the future work.

2 Preliminaries About NRS in IFOIS

As a structure that rigorously reflects the degree of favor and against of experts when judging things, the intuitionistic fuzzy set is worthy of research. what we will review first are some basic concepts about IFS and intuitionistic fuzzy ordered information system (IFOIS). In the light of the application of NRSs in IFOIS, we are about to review the NRSs and point out the shortcomings of NRSs.

Let U be a non-empty finite universe set, then an IFS in the universe U can be defined as

$$A = \{ < \mu_A(x), v_A(x) > \}, x \in U$$

where the functions $\mu_A : U \rightarrow [0,1]$, $v_A : U \rightarrow [0,1]$ represent the degree of membership and non-membership of x in A, and satisfy $0 \leq \mu_A(x) + v_A(x) \leq 1$. Furthermore, $\omega_A(x) = 1 - \mu_A(x) - v_A(x)$ denotes the hesitation degree that x belongs to A. If $\omega_A(x) = 0$, the intuitionistic fuzzy set degenerates into a classical fuzzy set.

For any IFS A, B, their related operator properties are as follows

(1) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x) \wedge v_A(x) = v_B(x), \forall x \in U$;
(2) $A \supseteq B \Leftrightarrow \mu_A(x) \geq \mu_B(x) \wedge v_A(x) \leq v_B(x), \forall x \in U$;
(3) $A \cap B = \{x \in U | \mu_A(x) \wedge \mu_B(x), v_A(x) \vee v_B(x)\}$;
(4) $A \cup B = \{x \in U | \mu_A(x) \vee \mu_B(x), v_A(x) \wedge v_B(x)\}$
(5) $A^C = \{x \in U | v_A(x), \mu_A(x)\}$.

An intuitionistic fuzzy information system(IFIS) is a triple $\widetilde{I} = (U, AT, F)$, where

$U = \{x_1, x_2, \ldots, x_m\}$ represents a non-empty, finite universe set.
$AT = (a_1, a_2, \ldots, a_n)$ is a set of conditional attributes.
$F = \bigcup_{a_j \in AT} f_{a_j}$, f_{a_j} is the domain of attribute a_j.

In IFIS $\widetilde{I} = (U, AT, F)$, for $\forall a \in AT$, and we define

$$f(x_i, a) \leq f(x_j, a) \Leftrightarrow (\forall\, a \in AT)[\mu_a(x_i) \leq \mu_a(x_j), v_a(x_i) \geq v_a(x_j)],$$

and

$$f(x_i, a) \geq f(x_j, a) \Leftrightarrow (\forall\, a \in AT)[\mu_a(x_i) \geq \mu_a(x_j), v_a(x_i) \leq v_a(x_j)].$$

They are respectively called increasing and decreasing partial order in IFIS. If the domain of an attribute is an increasing or decreasing partial order, then the attribute can be a criterion. Specifically, if all attributes are criteria, then IFIS is an intuitionistic fuzzy ordered information system (IFOIS), which is denoted as $\widetilde{I}^{\geq} = (U, AT, F)$. Furthermore, we consider $\widetilde{I}^{\geq} = (U, AT \cup \{d\}, F)$ as an intuitionistic fuzzy ordered decision information system (IFODIS), where the relation $R_d = \{(x_i, x_j) \in U \times U | f(x_i, d) = f(x_j, d)\}$ is an equivalence relation.

With a view to the difference in intuitionistic fuzzy datasets, the degree of difference varies between intuitionistic fuzzy objects. Hence, we provide a distance function to measure the distance between any two intuitionistic fuzzy numbers. For an IFOIS $\widetilde{I}^{\geq} = (U, AT, F)$, $A \subseteq AT$, $\forall x_i, x_j \in U$, the distance between object x_i and x_j on A can be given as follows:

$$\widehat{D}_A(x_i, x_j) = \left(\sum_{n=1}^{|A|} \left(\left| \mu_{a_n}(x_i) - \mu_{a_n}(x_j) \right|^q + \left| v_{a_n}(x_i) - v_{a_n}(x_j) \right|^q \right) \right)^{\frac{1}{q}}.$$

The distance $\widehat{D}_A(x_i, x_j)$ is named Manhattan distance when $q = 1$. $\widehat{D}_A(x_i, x_j)$ turns to be Euclidean distance when $q = 2$. Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, for $\forall x_i, x_j \in U$, $A \subseteq AT$, the neighborhood relation $\widetilde{R}_A^{\delta}$ on attribute A is defined as:

$$\widetilde{R}_A^{\delta} = \{(x_i, x_j) \in U \times U \big| \widehat{D}_A(x_i, x_j) \leq \delta\},$$

where neighborhood threshold $\delta gt0$. The neighborhood relation $\widetilde{R}_A^{\delta}$ satisfies reflexivity, symmetry and non-transitivity, and the corresponding neighborhood class of x_i induced by $\widetilde{R}_A^{\delta}$ is

$$\widetilde{[x_i]}_A^{\delta} = \{x_j \in U \big| (x_i, x_j) \in \widetilde{R}_A^{\delta}\},$$

where the neighborhood class $\widetilde{[x_i]}_A^{\delta}$ is an object set in which the distance between each object x_j and x_i is less than or equal to δ.

Definition 2.1 Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, for $\forall X \subseteq U$, $A \subseteq AT$, neighborhood threshold $\delta gt0$. The lower and upper approximations of X with respect to neighborhood relation $\widetilde{R}_A^{\delta}$ are defined as

$$\underline{\widetilde{R}_A^{\delta}}(X) = \{x \in U \big| \widetilde{[x]}_A^{\delta} \subseteq X\},$$
$$\overline{\widetilde{R}_A^{\delta}}(X) = \{x \in U \big| \widetilde{[x]}_A^{\delta} \cap X \neq \emptyset\}.$$

$\underline{\widetilde{R}_A^{\delta}}(X)$ and $\overline{\widetilde{R}_A^{\delta}}(X)$ are a pair of approximations operators. If $\underline{\widetilde{R}_A^{\delta}}(X) = \overline{\widetilde{R}_A^{\delta}}(X)$, then X is a definable set, otherwise X is rough. Three regions of X with respect to $\widetilde{R}_A^{\delta}$ can be obtained as $POS_A^{\delta}(X) = \underline{\widetilde{R}_A^{\delta}}(X)$, $NEG_A^{\delta}(X) = \sim \underline{\widetilde{R}_A^{\delta}}(X)$ and $BND_A^{\delta}(X) = \overline{\widetilde{R}_A^{\delta}}(X) - \underline{\widetilde{R}_A^{\delta}}(X)$.

The rough measure of NRS in IFOIS is similar to classical rough set. In IFOIS $\widetilde{I}^{\geq} = (U, AT, F)$, attribute subsets $A \subseteq AT$, for $\forall X \subseteq U$, the roughness of X with respect to neighborhood relation $\widetilde{R}_A^{\delta}$ is defined as

$$\rho(\widetilde{R}_A^{\delta}, X) = 1 - \frac{\left|\underline{\widetilde{R}_A^{\delta}}(X)\right|}{\left|\overline{\widetilde{R}_A^{\delta}}(X)\right|}.$$

Moreover, the approximation quality of D determined by conditional attribute A is called the degree of dependence. It can be denoted as

$$\gamma^{\delta}(A, D) = = \frac{\left|\sum_{i=1}^{n} \underline{\widetilde{R}_A^{\delta}}(D_i)\right|}{|U|},$$

where $R_d = \{(x_i, x_j) \in U \times U | f(x_i, d) = f(x_j, d)\}$, $U/d = \{D_1, D_2, \ldots, D_n\}$.

In RST, the classification and acquirement of datasets have been a research hotspot. Nevertheless, the NRS model has some shortcomings in the processing of ordered datasets, and the partial order relationship between samples in the NRS model has not been fully explored. Consequently, the relationship between the objects in the neighborhood class is not clear so that we cannot obtain the required set of eligible samples.

In real-life applications, there are many cases involving sample selections. For example, in the company's recruitment example mentioned in this paper, the interviewer selects people on the basis of the recruitment criteria. In the blind date market, blind daters select their spouses according to their criteria for mate selection. These problems are based on different criteria to select more qualified samples. If we mean to obtain more objects that meet the conditions, the NRS alone will be difficult to accomplish the goal. Accordingly, we will introduce a new model to effectively solve this problem.

3 Applications and Uncertainty Measures of NDRS in IFOIS

This section will revolve around a novel relation called the neighborhood dominance relation and center on practical applications of the model established on this relation.

3.1 Neighborhood Dominance Rough Sets

In the DRSA model, the dominance relation takes into account the ordering properties of criteria. For simplicity without loss of generality, we only consider the dominance relation by the increasing preference. Let us define the dominance relation in IFOIS. Given an IFOIS $\widetilde{I}^{\geq} = (U, AT, F)$, for $\forall x_i, x_j \in U$, $A \subseteq AT$, the dominance relation $\widetilde{R}_A^{\geq}$ on attribute A can be defined as

$$\widetilde{R}_A^{\geq} = \{(x_i, x_j) \in U \times U | (\forall a \in A)[\mu_a(x_i) \leq \mu_a(x_j), \nu_a(x_i) \geq \nu_a(x_j)]\}.$$

Combined the idea of the dominance relation and the neighborhood relation, the neighborhood dominance relation in IFOIS can be obtained. Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, for $\forall x_i, x_j \in U$, $A \subseteq AT$, $0 \leq \delta \leq 1$, the neighborhood dominance relation $\widetilde{R}^{\delta\geq}$ on attribute subset A is defined as

$$\begin{aligned}\widetilde{R}_A^{\delta\geq} = \{&(x_i, x_j) \in U \times U \Big| (\forall a \in A)\widehat{D}_A(x_i, x_j) \\ &\leq \delta \wedge [\mu_a(x_i) \leq \mu_a(x_j), \nu_a(x_i) \geq \nu_a(x_j)]\},\end{aligned}$$

where $\widetilde{R}_A^{\delta\geq}$ is the neighborhood dominance relation of the IFOIS. When $\delta \to \infty$, the intuitionistic fuzzy neighborhood dominance relation degenerates into traditional intuitionistic fuzzy dominance relation. It can be found that the neighborhood dominance relation not only describes the partial order relationship, but also characterizes the similarity between objects.

Let denote

$$\begin{aligned}\widetilde{[x_i]}_A^{\delta\geq} &= \{x_j \in U \Big| x_j \widetilde{R}_A^{\delta\geq} x_i\} \\ &= \{x_j \in U \Big| (\forall a \in A)\widehat{D}_A(x_i, x_j) \leq \delta \wedge [\mu_a(x_i) \\ &\leq \mu_a(x_j), \nu_a(x_i) \geq \nu_a(x_j)]\}, \\ \widetilde{[x_i]}_A^{\delta\leq} &= \{x_j \in U \Big| x_i \widetilde{R}_A^{\delta\geq} x_j\} \\ &= \{x_j \in U \Big| (\forall a \in A)\widehat{D}_A(x_i, x_j) \leq \delta \wedge [\mu_a(x_i) \\ &\geq \mu_a(x_j), \nu_a(x_i) \leq \nu_a(x_j)]\}, \\ U/\widetilde{R}_A^{\delta\geq} &= \{\widetilde{[x_i]}_A^{\delta\geq} | x_i \in U\},\end{aligned}$$

in which $i \in 1, 2, \ldots, |U|\}$, $\widetilde{[x_i]}_A^{\delta\geq}$ and $\widetilde{[x_i]}_A^{\delta\leq}$ represent the neighborhood dominance and dominated class of x_i, respectively. $U/\widetilde{R}_A^{\delta\geq}$ denotes a classification of U about $\widetilde{R}_A^{\delta\geq}$ in IFIS.

From the above-mentioned neighborhood dominance relation and its dominance class, we can obtain NDRS model in IFOIS.

Definition 3.1 Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS. For $X \subseteq U$, $A \subseteq AT$, the lower and upper approximations of X with respect to the neighborhood dominance relation $\widetilde{R}_A^{\delta\geq}$ are defined as follows

$$\underline{\widetilde{R}_A^{\delta\geq}}(X) = \left\{x \in U \Big| \widetilde{[x]}_A^{\delta\geq} \subseteq X)\right\},$$

$$\overline{\widetilde{R}_A^{\delta\geq}}(X) = \left\{x \in U \Big| \widetilde{[x]}_A^{\delta\geq} \cap X \neq \emptyset)\right\}.$$

From above definition, $\underline{\widetilde{R}_A^{\delta\geq}}(X)$ and $\overline{\widetilde{R}_A^{\delta\geq}}(X)$ are a pair of approximation operators. If $\underline{\widetilde{R}_A^{\delta\geq}}(X) = \overline{\widetilde{R}_A^{\delta\geq}}(X)$, then X is a definable set, otherwise X is called a rough set in IFIS. Three disjoint regions of X named positive region, negative region and boundary region under the intuitionistic fuzzy neighborhood dominance relation $\widetilde{R}_A^{\delta\geq}$ are denoted as $POS(X) = \underline{\widetilde{R}_A^{\delta\geq}}(X)$, $NEG(X) = \sim\underline{\widetilde{R}_A^{\delta\geq}}(X)$, and $BND(X) = \overline{\widetilde{R}_A^{\delta\geq}}(X) - \underline{\widetilde{R}_A^{\delta\geq}}(X)$.

When we use the NDRS to handle ordered datasets, the partial order relationship between the samples is considered in the neighborhood class, which achieves the purpose of classifying ordinal datasets and further explores the relationship between the samples. As shown in Fig. 4, it is assumed that x_1 is a standard object that meets the conditions of the problem, marked in blue. Meanwhile, the objects marked in yellow and green respectively represent the completely unqualified objects and partially qualified objects. When we classify the dataset by the neighborhood relation and neighborhood dominance relation to obtain more qualified objects similar to x_1, we are able to find that both qualified, unqualified and partially qualified objects are included in the neighborhood class. Conversely, the neighborhood dominance class contains all the eligible objects that meet the criteria.

From a mathematical point of view, the stricter relationships will inevitably lead to the finer divisions in the dataset. So for the same target set X, $B \subseteq AT$, we have $\widetilde{[x]}_B^{\delta\geq} \subseteq \widetilde{[x]}_B^{\delta}$ due to excluding unqualified samples, then it apparently comes to a conclusion that $\underline{\widetilde{R}_B^{\delta}}(X) \subseteq \underline{\widetilde{R}_B^{\delta\geq}}(X)$ and $\overline{\widetilde{R}_B^{\delta\geq}}(X) \subseteq \overline{\widetilde{R}_B^{\delta}}(X)$. When faced with the different needs of decision-makers, we can select corresponding eligible samples from the dataset. Therefore, the NDRS model can repair the shortcomings of NRS model, and the roughness of the NDRS is lower than NRS during conceptual approximation.

Proposition 3.1 *Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, attribute subsets $B \subseteq AT$, $0 \leq \delta_1 \leq \delta_2 \leq 1$, for $\forall X \subseteq U$, then we have*

(1) $\underline{\widetilde{R}_B^{\delta}}(X) \subseteq \underline{\widetilde{R}_B^{\delta\geq}}(X) \subseteq \underline{\widetilde{R}_{AT}^{\delta\geq}}(X)$,
$\overline{\widetilde{R}_B^{\delta}}(X) \supseteq \overline{\widetilde{R}_B^{\delta\geq}}(X) \supseteq \overline{\widetilde{R}_{AT}^{\delta\geq}}(X)$;

(2) If $\widetilde{R}_B^{\delta\geq} = \widetilde{R}_{AT}^{\delta\geq}$, then
$\underline{\widetilde{R}_A^{\delta\geq}}(X) = \underline{\widetilde{R}_{AT}^{\delta\geq}}(X), \overline{\widetilde{R}_B^{\delta\geq}}(X) = \overline{\widetilde{R}_{AT}^{\delta\geq}}(X)$.

(3) $\underline{\widetilde{R}_B^{\delta_2}}(X) \subseteq \underline{\widetilde{R}_B^{\delta_2\geq}}(X) \subseteq \underline{\widetilde{R}_B^{\delta_1\geq}}(X)$,
$\overline{\widetilde{R}_B^{\delta_1\geq}}(X) \subseteq \overline{\widetilde{R}_B^{\delta_2\geq}}(X) \subseteq \overline{\widetilde{R}_B^{\delta_2}}(X)$.

Proof

(1) For $\forall X \subseteq U$, $B \subseteq AT$, we have $\widetilde{[x]}_{AT}^{\delta\geq} \subseteq \widetilde{[x]}_B^{\delta\geq} \subseteq \widetilde{[x]}_B^{\delta} \subseteq X$. Thus $\underline{\widetilde{R}_B^{\delta}}(X) \subseteq \underline{\widetilde{R}_B^{\delta\geq}}(X) \subseteq \underline{\widetilde{R}_{AT}^{\delta\geq}}(X)$. Otherwise, $(\widetilde{[x]}_{AT}^{\delta\geq} \cap X) \subseteq (\widetilde{[x]}_B^{\delta\geq} \cap X) \subseteq (\widetilde{[x]}_B^{\delta} \cap X)$, thus $\overline{\widetilde{R}_B^{\delta}}(X) \supseteq \overline{\widetilde{R}_B^{\delta\geq}}(X) \supseteq \overline{\widetilde{R}_{AT}^{\delta\geq}}(X)$.

(2) From $\widetilde{R}_B^{\delta\geq} = \widetilde{R}_{AT}^{\delta\geq}$, $\widetilde{[x]}_B^{\delta\geq} = \widetilde{[x]}_{AT}^{\delta\geq}$. Then $\underline{\widetilde{R}_A^{\delta\geq}}(X) = \underline{\widetilde{R}_{AT}^{\delta\geq}}(X)$ and $\overline{\widetilde{R}_B^{\delta\geq}}(X) = \overline{\widetilde{R}_{AT}^{\delta\geq}}(X)$ can be proved directly.

(3) Since $0 \leq \delta_1 \leq \delta_2 \leq 1$, then $\widetilde{[x]}_B^{\delta_1\geq} \subseteq \widetilde{[x]}_B^{\delta_2\geq} \subseteq \widetilde{[x]}_B^{\delta_2}$. Hence, $\underline{\widetilde{R}_B^{\delta_2}}(X) \subseteq \underline{\widetilde{R}_B^{\delta_2\geq}}(X) \subseteq \underline{\widetilde{R}_B^{\delta_1\geq}}(X)$, $\overline{\widetilde{R}_B^{\delta_1\geq}}(X) \subseteq \overline{\widetilde{R}_B^{\delta_2\geq}}(X) \subseteq \overline{\widetilde{R}_B^{\delta_2}}(X)$ can be proved.

In the process of approximating concepts through rough sets, there are two important factors, the generation of the class and the determination of target sets. Many people pay attention to the generation method of the class, but ignore the determination of the target set. However, in the face of practical problems, the selection of the target set is often directly related to the approximate results of the problem.

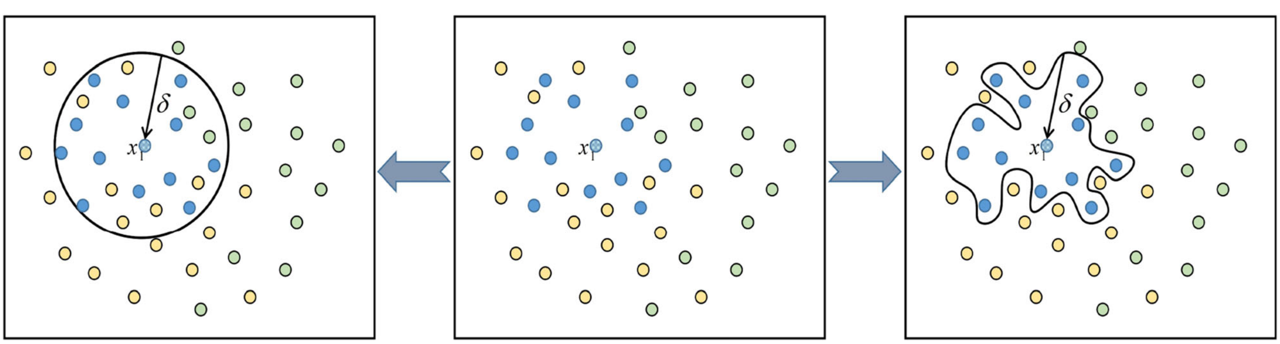


Fig. 4 The detection of objects between NRS and NDRS

When we select an object that meets the conditions of the problem as the target set, its approximate result can better meet the needs of decision-makers. On the contrary, when we select an object that does not meet the conditions of the problem as the target set, its approximate results are harder to meet the conditions of the actual problem. Therefore, during conceptual approximation by rough set, how to determine an optimal set of objects that meets the conditions of the problem is an important issue. Given an information system $\widetilde{I}^{\geq} = (U, AT, F)$, according to the different needs of the problem, we can set a standard object x^* that meets the conditions of the problem. Then the optimal set named the target set X can be obtained by $X = \widetilde{[x^*]}_{AT}^{\delta\geq} - \{x^*\}$.

3.2 Uncertainty Measures of NDRS

Due to the existence of boundary region, there exists uncertainty in rough set. The rough measure of NDRS is similar to the classical rough set. In order to measure the uncertainty of NDRS in IFOIS, the roughness is defined as follows.

Definition 3.2 Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, and $B \subseteq AT$, for $\forall X \subseteq U$. The rough measure of X under the neighborhood dominance relation $\widetilde{R}_B^{\delta\geq}$ is defined as

$$\rho(\widetilde{R}_B^{\delta\geq}, X) = 1 - \frac{\left|\underline{\widetilde{R}_B^{\delta\geq}}(X)\right|}{\left|\overline{\widetilde{R}_B^{\delta\geq}}(X)\right|}.$$

The roughness $\rho(\widetilde{R}_B^{\delta\geq}, X)$ is used to reflect the degree of incomplete knowledge of set X. When $\overline{\widetilde{R}_B^{\delta\geq}}(X) = \emptyset$, it is obvious that $\rho(\widetilde{R}_B^{\delta\geq}, X) = 1$. Based on the roughness of X, the accuracy of X is $\alpha(\widetilde{R}_B^{\delta\geq}, X) = 1 - \rho(\widetilde{R}_B^{\delta\geq}, X)$

Definition 3.3 Given an IFOIS $\widetilde{I}^{\geq} = (U, AT \cup d, F)$, $B \subseteq AT$, the approximate quality of decision attribute d determined by conditional attribute B is called the degree of dependency, which is defined as

$$\gamma^{\delta\geq}(B, D) = \frac{1}{|U|} \sum_{i=1}^{n} \underline{\widetilde{R}_B^{\delta\geq}}(D_i).$$

where $\widetilde{R}_d^{\geq} = \{(x_i, x_j) \in U \times U | f(x_i, d) = f(x_j, d)\}, U/d = \{D_1, D_2, \ldots, D_m\}$.

Proposition 3.2 Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, and $B \subseteq AT$. For $\forall X \subseteq U$, then the following results hold

(1) $0 \leq \rho(\widetilde{R}_B^{\delta\geq}, X) \leq \rho(\widetilde{R}_B^{\delta}, X) \leq 1$,
$0 \leq \gamma^{\delta}(B, D) \leq \gamma^{\delta\geq}(B, D) \leq 1$;

(2)
$$\rho(\widetilde{R}_B^{\delta\geq}, X) = 1 - \frac{\left|\underline{\widetilde{R}_B^{\delta\geq}}(X)\right|}{\left|\overline{\widetilde{R}_B^{\delta\geq}}(X)\right|} = 1 - \frac{\left|\underline{\widetilde{R}_B^{\delta\geq}}(X)\right|}{|U| - \left|\underline{\widetilde{R}_B^{\delta\geq}}(\sim X)\right|},$$
$$\rho(\widetilde{R}_B^{\delta}, X) = 1 - \frac{\left|\underline{\widetilde{R}_B^{\delta}}(X)\right|}{\left|\overline{\widetilde{R}_B^{\delta}}(X)\right|} = 1 - \frac{\left|\underline{\widetilde{R}_B^{\delta}}(X)\right|}{|U| - \left|\underline{\widetilde{R}_B^{\delta}}(\sim X)\right|};$$

(3) If $\widetilde{R}_B^{\delta\geq} = \widetilde{R}_{AT}^{\delta\geq}$, then $\rho(\widetilde{R}_B^{\delta\geq}, X) = \rho(\widetilde{R}_{AT}^{\delta\geq}, X)$, $\gamma^{\delta\geq}(B, D) = \gamma^{\delta\geq}(AT, D)$;

(4) If $A \subseteq AT$, then $\rho(\widetilde{R}_{AT}^{\delta\geq}, X) \leq \rho(\widetilde{R}_A^{\delta\geq}, X) \leq \rho(\widetilde{R}_A^{\delta}, X)$, $\gamma^{\delta\geq}(AT, D) \geq \gamma^{\delta\geq}(B, D) \geq \gamma^{\delta}(B, D)$.

Proof

(1) The proof can be directly obtained by Definition 3.2 and Proposition 3.1.

(2) For $\forall x \in \underline{\widetilde{R}_B^{\delta\geq}}(\sim X)$, we have $\forall x \in \underline{\widetilde{R}_B^{\delta\geq}}(\sim X) \Leftrightarrow \widetilde{[x]}_B^{\delta\geq} \subseteq \sim X \Leftrightarrow \widetilde{[x]}_B^{\delta\geq} \cap X = \emptyset \Leftrightarrow x \notin \overline{\widetilde{R}_B^{\delta\geq}}(X) \Leftrightarrow x \in \sim \overline{\widetilde{R}_B^{\delta\geq}}(X)$. Therefore, $\left|\overline{\widetilde{R}_B^{\delta\geq}}(X)\right| = |U| - \left|\underline{\widetilde{R}_B^{\delta\geq}}(\sim X)\right|$. The proof of $\widetilde{R}_B^{\delta\geq}$ is analogous.

(3) If $\widetilde{R}_B^{\delta\geq} = \widetilde{R}_{AT}^{\delta\geq}$, then $\underline{\widetilde{R}_A^{\delta\geq}}(X) = \underline{\widetilde{R}_{AT}^{\delta\geq}}(X)$ and $\overline{\widetilde{R}_B^{\delta\geq}}(X) = \overline{\widetilde{R}_{AT}^{\delta\geq}}(X)$ can be obtained according to Propo sition 3.1. Thus, $\rho(\widetilde{R}_B^{\delta\geq}, X) = \rho(\widetilde{R}_{AT}^{\delta\geq}, X)$ and $\gamma^{\delta\geq}(B, D) = \gamma^{\delta\geq}(AT, D)$ can be proved.

(4) Due to $A \subseteq AT$, we can have $\underline{\widetilde{R}_A^{\delta}}(X) \subseteq \underline{\widetilde{R}_A^{\delta\geq}}(X) \subseteq \underline{\widetilde{R}_{AT}^{\delta\geq}}(X)$, and $\overline{\widetilde{R}_A^{\delta}}(X) \supseteq \overline{\widetilde{R}_A^{\delta\geq}}(X) \supseteq \overline{\widetilde{R}_{AT}^{\delta\geq}}(X)$. There fore, $\rho(\widetilde{R}_{AT}^{\delta\geq}, X) \leq \rho(\widetilde{R}_B^{\delta\geq}, X) \leq \rho(\widetilde{R}_A^{\delta\geq}, X)$ and $\gamma^{\delta\geq}(AT, D) \geq \gamma^{\delta\geq}(B, D) \geq \gamma^{\delta}(B, D)$ can be proved.

Example 3.1 As we all know, there are all kinds of talent recruitments every year. Excellent talents are always favored by enterprises, but mediocre talents face very huge competitive pressure when applying. Only by showing their own strengths can they stand out in the selection process.

Here is an actual case of a company called Alibaba when interviewing talents, Table 2 is an IFOIS $\widetilde{I}^{\geq} = (U, AT, F)$ about talent interview evaluation, and the intuitionistic fuzzy value in Table 2 is given by the interviewer according to the performance of the candidates. Moreover, ten candidates in this interview can be regarded as ten objects in $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$. The evaluation conditions which can be regarded as conditional attributes $AT = \{a_1, a_2, a_3, a_4, a_5\}$ are divided into "hard conditions" and "soft performance". "Hard conditions" $A = \{a_1, a_2, a_3\}$ is composed of attributes a_1, a_2 and a_3, where a_1, a_2 and a_3 respectively represent written test results, educational background and work experience; "soft performance" $B = \{a_4, a_5\}$ consists of attributes a_4, a_5, where a_4, a_5 respectively mean language expression, management ability. Decision attribute d is the result of the interviewer's hiring. Y expresses that the interviewer is hired, N denotes that the interviewer is not hired. The universe U is partitioned by the decision attribute, which can be denoted as $U/d = \{D_Y, D_N\}$.

The company's recruitment requirement is a written test score of at least 70, preferably a graduate degree or above, and three to five years of work experience, and good expression and management skills. It is assumed that meeting the recruitment conditions requires at least $\mu(x_i) \geq 0.7$ and $v(x_i) \leq 0.2$ under conditional attributes. Hence, we set the standard object x^* whose membership degree $\mu(x^*) \geq 0.7$ and non-membership degree $v(x^*) \leq 0.2$. Suppose $\delta = 0.6$.

Table 2 Intuitionistic fuzzy ordered information system

U	a_1	a_2	a_3	a_4	a_5	d
x_1	(0.7,0.2)	(0.7,0.2)	(0.6,0.2)	(0.7,0.2)	(0.7,0.2)	N
x_2	(0.7,0.2)	(0.7,0.2)	(0.4,0.3)	(0.7,0.2)	(0.6,0.2)	N
x_3	(0.6,0.2)	(0.6,0.3)	(0.4,0.2)	(0.7,0.2)	(0.7,0.3)	N
x_4	(0.8,0.1)	(0.8,0.1)	(0.8,0.2)	(0.7,0.2)	(0.8,0.2)	Y
x_5	(0.7,0.1)	(0.7,0.1)	(0.8,0.2)	(0.8,0.2)	(0.7,0.2)	Y
x_6	(0.8,0.1)	(0.8,0.1)	(0.8,0.2)	(0.8,0.2)	(0.7,0.2)	Y
x_7	(0.5,0.3)	(0.7,0.2)	(0.5,0.3)	(0.6,0.3)	(0.6,0.2)	N
x_8	(0.6,0.2)	(0.7,0.2)	(0.6,0.3)	(0.6,0.2)	(0.6,0.1)	N
x_9	(0.8,0.2)	(0.8,0.2)	(0.7,0.0)	(0.7,0.2)	(0.8,0.2)	Y
x_{10}	(0.8,0.2)	(0.9,0.1)	(0.7,0.2)	(0.7,0.2)	(0.7,0.2)	Y

It is assumed that the conditional attributes A and AT correspond to the situation when recruitment requirements decrease and increase. The generated neighborhood classes with respect to $A = \{a_1, a_2, a_3\}$ and $AT = \{a_1, a_2, a_3, a_4, a_5\}$ are shown in Table 3.

From the neighborhood classes induced by A and AT, we can verify that the neighborhood information granules formed by AT is finer than that formed by A. Then we compute the neighborhood dominance classes with respect to $A = \{a_1, a_2, a_3\}$ and $AT = \{a_1, a_2, a_3, a_4, a_5\}$ in Table 3.

In order to determine the optimal set that meets different conditions, we will discuss it in the following different situations.

(1) If all conditions are taken into account, then the target set X_1 can be determined by the neighborhood dominance class of the standard object x^* under AT, which can be denoted as $X_1 = \widetilde{[x^*]}_{AT}^{\delta\geq} - \{x^*\} = \{x_4, x_5, x_6, x_9, x_{10}\}$. Then the target set X_1 is a set of objects that meet all recruitment conditions.

(2) If we only consider "hard conditions" $A = \{a_1, a_2, a_3\}$ as the recruitment requirements, then the target set X_2 can be expressed as $X_2 = \widetilde{[x^*]}_{A}^{\delta\geq} - \{x^*\} = \{x_4, x_5, x_6, x_9, x_{10}\}$. We know that X_2 is a set of objects that meet "hard conditions".

The process of determining the optimal set not only improves the accuracy of the approximate set, but also makes the approximate result better meet the needs of decision-makers.

We have acquired neighborhood classes under different recruitment conditions. According to Definition 2.1, we can compute the lower and upper approximations of X under attributes A and AT based on the NRS, which are

$$\underline{\widetilde{R}_{AT}^{\delta}}(X_1) = \{x_4, x_6\}; \qquad \underline{\widetilde{R}_{A}^{\delta}}(X_2) = \emptyset;$$

$$\overline{\widetilde{R}_{AT}^{\delta}}(X_1) = \{x_1, x_4, x_5, x_6, x_9, x_{10}\}. \qquad \overline{\widetilde{R}_{A}^{\delta}}(X_2) = \{x_1, x_4, x_5, x_6, x_8, x_9, x_{10}\}.$$

Table 3 Neighborhood classes and neighborhood dominance classes

U	$\widetilde{[x]}_{AT}^{\delta}$	$\widetilde{[x]}_{A}^{\delta}$	$\widetilde{[x]}_{AT}^{\delta\geq}$	$\widetilde{[x]}_{A}^{\delta\geq}$
x_1	$\{x_1, x_2, x_3, x_5, x_8, x_9, x_{10}\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$	$\{x_1, x_5, x_9, x_{10}\}$	$\{x_1, x_4, x_5, x_6, x_9, x_{10}\}$
x_2	$\{x_1, x_2, x_3, x_7, x_8\}$	$\{x_1, x_2, x_3, x_7, x_8\}$	$\{x_1, x_2\}$	$\{x_1, x_2\}$
x_3	$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_3, x_7, x_8\}$	$\{x_1, x_3\}$	$\{x_1, x_3\}$
x_4	$\{x_4, x_5, x_6, x_9, x_{10}\}$	$\{x_1, x_4, x_5, x_6, x_9, x_{10}\}$	$\{x_4\}$	$\{x_4, x_6\}$
x_5	$\{x_1, x_4, x_5, x_6, x_{10}\}$	$\{x_1, x_4, x_5, x_6, x_8, x_{10}\}$	$\{x_5, x_6\}$	$\{x_4, x_5, x_6\}$
x_6	$\{x_4, x_5, x_6, x_{10}\}$	$\{x_1, x_4, x_5, x_6, x_9, x_{10}\}$	$\{x_6\}$	$\{x_4, x_6\}$
x_7	$\{x_2, x_7, x_8\}$	$\{x_1, x_2, x_3, x_7, x_8\}$	$\{x_7, x_8\}$	$\{x_1, x_7, x_8\}$
x_8	$\{x_1, x_2, x_7, x_8\}$	$\{x_1, x_2, x_3, x_5, x_7, x_8\}$	$\{x_8\}$	$\{x_1, x_5, x_8\}$
x_9	$\{x_1, x_4, x_9, x_{10}\}$	$\{x_1, x_4, x_6, x_9, x_{10}\}$	$\{x_9\}$	$\{x_9\}$
x_{10}	$\{x_1, x_4, x_5, x_6, x_9, x_{10}\}$	$\{x_1, x_4, x_5, x_6, x_9, x_{10}\}$	$\{x_{10}\}$	$\{x_{10}\}$

When we consider all conditions, it is obvious that x_4, x_6 will certainly be hired, and $x_1, x_4, x_5, x_6, x_9, x_{10}$ will possibly be hired. When we only consider "hard conditions", we can obtain that no one will definitely be hired, and $x_1, x_4, x_5, x_6, x_8, x_9, x_{10}$ may be hired.

The neighborhood dominance classes can be shown in Table 3. According to Definition 3.1, we can obtain the lower and upper approximations of X based on the NDRS under attributes A and AT.

$$\underline{\widetilde{R}_{AT}^{\delta\geq}}(X_1) = \{x_4, x_5, x_6, x_9, x_{10}\}; \qquad \underline{\widetilde{R}_{A}^{\delta\geq}}(X_2) = \{x_4, x_5, x_6, x_9, x_{10}\};$$
$$\overline{\widetilde{R}_{AT}^{\delta\geq}}(X_1) = \{x_1, x_4, x_5, x_6, x_9, x_{10}\}. \qquad \overline{\widetilde{R}_{A}^{\delta\geq}}(X_2) = \{x_1, x_4, x_5, x_6, x_9, x_{10}\}.$$

When all conditions are considered, we can know that $x_4, x_5, x_6, x_9, x_{10}$ will surely be hired, and x_1, x_4, x_5, x_6, x_9 $, x_{10}$ will possibly be hired. When only the "hard conditions" are considered, we can find that $x_4, x_5, x_6, x_9, x_{10}$ will certainly be hired, and $x_1, x_4, x_5, x_6, x_9, x_{10}$ will possibly be hired.

The approximate result displays that the accuracy of the approximate sets is greatly improved. Besides, NDRS can solve the common problem of selecting more qualified samples according to the criteria in life. We use rough measures to prove the superiority of NDRS in this kind of problems.

The roughness of NRS and NDRS can be obtained under attributes AT and A

$$\rho(\widetilde{R}_{AT}^{\delta}, X_1) = 1 - \frac{\left|\underline{\widetilde{R}_{AT}^{\delta}}(X_1)\right|}{\left|\overline{\widetilde{R}_{AT}^{\delta}}(X_1)\right|} = \frac{4}{6}, \qquad \rho(\widetilde{R}_{A}^{\delta}, X_2) = 1 - \frac{\left|\underline{\widetilde{R}_{A}^{\delta}}(X_2)\right|}{\left|\overline{\widetilde{R}_{A}^{\delta}}(X_2)\right|} = 1.$$
$$\rho(\widetilde{R}_{AT}^{\delta\geq}, X_1) = 1 - \frac{\left|\underline{\widetilde{R}_{AT}^{\delta\geq}}(X_1)\right|}{\left|\overline{\widetilde{R}_{AT}^{\delta\geq}}(X_1)\right|} = \frac{1}{6}, \qquad \rho(\widetilde{R}_{A}^{\delta\geq}, X_2) = 1 - \frac{\left|\underline{\widetilde{R}_{A}^{\delta\geq}}(X_2)\right|}{\left|\overline{\widetilde{R}_{A}^{\delta\geq}}(X_2)\right|} = \frac{1}{6}.$$

So we can have

$$\rho(\widetilde{R}_{AT}^{\delta\geq}, X_1) \leq \rho(\widetilde{R}_{AT}^{\delta}, X_1), \qquad \rho(\widetilde{R}_{A}^{\delta\geq}, X_2) \leq \rho(\widetilde{R}_{A}^{\delta}, X_2).$$

Since $D_Y = \{x_4, x_5, x_6, x_9, x_{10}\}$, $D_N = \{x_1, x_2, x_3, x_7, x_8\}$. Then the dependence degree of NRS and NDRS can be obtained under attributes AT and A in the following.

$$\gamma^{\delta}(AT, D) = \frac{\sum_{i=1}^{n} \underline{\widetilde{R}_{AT}^{\delta}}(D_i)}{|U|} = \frac{6}{10}, \qquad \gamma^{\delta}(A, D) = \frac{\sum_{i=1}^{n} \underline{\widetilde{R}_{A}^{\delta}}(D_i)}{|U|} = \frac{3}{10}.$$
$$\gamma^{\delta\geq}(AT, D) = \frac{\sum_{i=1}^{n} \underline{\widetilde{R}_{AT}^{\delta\geq}}(D_i)}{|U|} = \frac{9}{10}, \qquad \gamma^{\delta\geq}(A, D) = \frac{\sum_{i=1}^{n} \underline{\widetilde{R}_{A}^{\delta\geq}}(D_i)}{|U|} = \frac{8}{10}.$$

Furthermore, we can obtain

$$\gamma^{\delta}(AT, D) \leq \gamma^{\delta\geq}(AT, D), \qquad \gamma^{\delta}(A, D) \leq \gamma^{\delta\geq}(A, D).$$

The results have verified that the roughness of the NRS is higher than that of the NDRS, and the dependence degree of NRS is lower than NDRS. Therefore, it is more reasonable to use the NDRS to process the intuitionistic fuzzy dataset to obtain the needed samples than the NRS.

From Example 3.1, it can be observed that only the case of depicting the target concept at a single granulation is discussed for IFOIS. However, in real life applications, we are faced with problems with different conditions and numerous properties. In view of the superiority of multigranulation rough set methods in solving practical problems and the complementarity of intuitionistic fuzzy set and multigranulation rough set methods, it is essential to combine multigranulation methods in NDRS of IFOIS to investigate the MDRS model. Meanwhile, the combination of MGRS has great application value in the financial risk analysis, specific sample extraction and talent selection.

4 Multigranulation Neighborhood Dominance Rough Sets in IFOIS

As for the Example 3.1, when talents are scarce and it is necessary to strive for talent resources in the recruitment process of company, then the interviewer may be required to meet at least one condition before being hired. When the talent is abundant and need to be selected from it, then the interviewer who meets all conditions can be hired. This phenomenon is exactly in line with the idea of MGRS. In order to more broadly apply NDRS the above problem, we introduce optimistic multigranulation neighborhood dominance rough sets (OMNDRS) and pessimistic multigranulation neighborhood dominance rough sets (PMNDRS).

4.1 Optimistic Multigranulation Neighborhood Dominance Rough Sets in IFOIS

This section will introduce OMNDRS using multiple neighborhood dominance relations in IFOIS, and discuss applications and properties of lower and upper approximation operators of OMNDRS.

Definition 4.1 Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, attribute subsets $A_1, A_2, \ldots, A_n \in AT (n \leq 2^{|AT|})$, the relation $\widetilde{R}_{A_i}^{\delta\geq}$ represents neighborhood dominance relations on attribute subsets $A_i(i = 1,2,\ldots,n)$. $\forall X \in P(U)$, $\widetilde{[x]}_{A_i}^{\delta\geq} = \{y \mid (x,y) \in \widetilde{R}_{A_i}^{\delta\geq}\}$ are called the i-th neighborhood dominance class which contains x with respect to the i-th neighborhood dominance relation $\widetilde{R}_{A_i}^{\delta\geq}$. The optimistic multigranulation lower and upper approximations of X are defined as follows:

$$\underline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^{n} A_i}}(X) = \left\{x \in U \,\middle|\, \bigvee_{i=1}^{n} (\widetilde{[x]}_{A_i}^{\delta\geq} \subseteq X)\right\},$$

$$\overline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^{n} A_i}}(X) = \left\{x \in U \,\middle|\, \bigwedge_{i=1}^{n} (\widetilde{[x]}_{A_i}^{\delta\geq} \cap X \neq \emptyset)\right\},$$

where logical operations "$\vee$" and "$\wedge$" mean "or" and "and", respectively. $\underline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^{n} A_i}}(X)$ and $\overline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^{n} A_i}}(X)$ are called optimistic multigranulation lower and upper approximation operators. If $\underline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^{n} A_i}}(X) \neq \overline{\widetilde{OM}_{\sum_{i=1}^{n}} A_i}^{\delta\geq}(X)$, then X is a rough set with respect to neighborhood dominance relations $\widetilde{R}_{A_i}^{\delta\geq}$ $(i = 1,2,\ldots,n)$, otherwise, it is a definable set. The three disjoint regions of the target set X are respectively $POS(X) = \underline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^{n} A_i}}(X)$, $NEG(X) = \sim \overline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^{n} A_i}}(X)$ and $BND(X) = \overline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^{n} A_i}}(X) - \underline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^{n} A_i}}(X)$.

Example 4.1 (Continued from Example 3.1) There are many uncertain factors in the annual recruitment, such as epidemic reasons, financial crisis, war factors, etc. These factors could cause a large number of unemployment every year and increase the difficulty of employment. Therefore, this case can be regarded as the pessimistic situation. However, there are still some economically developed countries and regions. Due to existence of economic prosperity, small population and technical levels, the difficulty of employment is reduced. Thus, this kind of circumstance can be considered as the optimistic situation. How can we use the rough set model to handle recruitment when facing the optimistic situation or the pessimistic situation.

According to the new recruitment conditions given by the company, the interviewers often prefer to hire people who meet these conditions:

Condition 1 Not only the "hard conditions" are passed, but also the language expression is better.
Condition 2 Not only the "hard conditions" are passed, but also the management ability is better.

From conditions 1 and 2, we can get two dominance relations named R_1, R_2 and their related dominance classes, when $\delta = 0.6$.

$$R_1 = \begin{pmatrix} 1&0&0&1&1&0&0&0&1&1\\ 1&1&0&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&1&0&0\\ 1&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

$$R_2 = \begin{pmatrix} 1&0&0&1&1&0&0&0&1&1\\ 1&1&0&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&1&0&0\\ 1&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

Take the target set $X = \widetilde{[x^*]}_{R_1}^{\delta\geq} - \{x^*\} = \widetilde{[x^*]}_{R_2}^{\delta\geq} - \{x^*\} = \{x_4, x_5, x_6, x_9, x_{10}\}$. Considering condition 1 and condition 2 respectively, we can obtain

$$\underline{\widetilde{R}_1^{\delta\geq}}(X) = \{x_4, x_5, x_6, x_9, x_{10}\}; \quad \underline{\widetilde{R}_2^{\delta\geq}}(X) = \{x_4, x_5, x_6, x_9, x_{10}\};$$
$$\overline{\widetilde{R}_1^{\delta\geq}}(X) = \{x_1, x_4, x_5, x_6, x_9, x_{10}\}. \quad \overline{\widetilde{R}_2^{\delta\geq}}(X) = \{x_1, x_4, x_5, x_6, x_9, x_{10}\}.$$

According to condition 1, we can obtain that $x_4, x_5, x_6, x_9, x_{10}$ will surely be employed, and $x_1, x_4, x_5, x_6, x_9, x_{10}$ will possibly employed. From condition 2, we can know $x_4, x_5, x_6, x_9, x_{10}$ must be hired, however, candidates $x_1, x_4, x_5, x_6, x_9, x_{10}$ may be hired.

In the optimistic situation, it is possible that interviewers are not good at expressing or his management ability needs to be improved when the "hard conditions" are passed, such talents can be cultivated after entry. There is aslo a situation that candidates are excellent in all aspects. So we raise two questions.

Question 1 If the company requires the candidate to meet at least one of the conditions, who will surely be hired.
Question 2 If the company requires the candidate to meet all conditions, who will possibly be hired.

According to Definition 4.1, we can obtain

$$\underline{\widetilde{OM}_{1+2}^{\delta\geq}}(X) = \{x_4, x_5, x_6, x_9, x_{10}\}; \quad \underline{\widetilde{OM}_{1+2}^{\delta\geq}}(X) = \underline{\widetilde{R}_1^{\delta\geq}}(X) \cup \underline{\widetilde{R}_2^{\delta\geq}}(X);$$
$$\overline{\widetilde{OM}_{1+2}^{\delta\geq}}(X) = \{x_1, x_4, x_5, x_6, x_9, x_{10}\}. \quad \overline{\widetilde{OM}_{1+2}^{\delta\geq}}(X) = \overline{\widetilde{R}_1^{\delta\geq}}(X) \cap \overline{\widetilde{R}_2^{\delta\geq}}(X).$$

From the above approximation results, we can know x_5, x_6, x_9, x_{10} must be employed when the company consider at least one condition, and $x_3, x_5, x_6, x_8, x_9, x_{10}$ may be employed when both conditions are considered. Besides, the properties of OMNDRS can be obtained.

Proposition 4.1 *Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, $A_i \subseteq AT(i = 1, 2, \ldots, n, n \leq 2^{|AT|})$, $0 \leq \delta_1 \leq \delta_2 \leq 1$, then*

(1) $\underline{\widetilde{OM}}^{\delta_2\geq}_{\sum_{i=1}^n A_i}(X) \subseteq \underline{\widetilde{OM}}^{\delta_1\geq}_{\sum_{i=1}^n A_i}(X)$,

(2) $\overline{\widetilde{OM}}^{\delta_1\geq}_{\sum_{i=1}^n A_i}(X) \subseteq \overline{\widetilde{OM}}^{\delta_2\geq}_{\sum_{i=1}^n A_i}(X)$.

Proposition 4.2 *Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, $A_i \subseteq AT(i = 1, 2, \ldots, n, n \leq 2^{|AT|})$. The optimistic multigranulation lower and upper approximations under IFOIS have following properties*

$(OL_1)\underline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(X) \subseteq X$; (Contraction)

$(OU_1)\overline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(X) \supseteq X$; (Extension)

$(OL_2)\underline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(\sim X) = \sim \overline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(X)$; (Duality)

$(OU_2)\overline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(\sim X) = \sim \underline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(X)$; (Duality)

$(OL_3)\underline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(\emptyset) = \emptyset$; (Normality)

$(OU_3)\overline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(\emptyset) = \emptyset$; (Normality)

$(OL_4)\underline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(U) = U$; (Co-normality)

$(OU_4)\overline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(U) = U$; (Co-normality)

$(OL_5)\underline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(X \cap Y) \subseteq \underline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(X) \cap \underline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(Y)$; (L-multiplication)

$(OU_5)\overline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(X \cup Y) \supseteq \overline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(X) \cup \overline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(Y)$; (L-addition)

$(OL_6)X \subseteq Y \Rightarrow \underline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(X) \subseteq \underline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(Y)$; (Granularity)

$(OU_6)X \subseteq Y \Rightarrow \overline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(X) \subseteq \overline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(Y)$; (Granularity)

$(OL_7)\underline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(X \cup Y) \supseteq \underline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(X) \cup \underline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(Y)$; (U-addition)

$(OU_7)\overline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(X \cap Y) \subseteq \overline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(X) \cap \overline{\widetilde{OM}}^{\delta\geq}_{\sum_{i=1}^n A_i}(Y)$. ($U$-multiplication)

Proof For a better explanation, we will give the proof of the properties under two neighborhood dominance relations. Assume $n = 2, A, B \subseteq AT$, when $A = B$, the above properties obviously hold. Thus, when $A \neq B$, then

(OL_1) For $\forall x \in \underline{\widetilde{OM}}^{\delta\geq}_{A+B}(X)$, according to Definition 4.1, we have $\widetilde{[x]}_A^{\delta\geq} \subseteq X$ or $\widetilde{[x]}_B^{\delta\geq} \subseteq X$. Since the neighborhood dominance relation $\widetilde{R}_A^{\delta\geq}$ satisfies reflexivity, then $x \in \widetilde{[x]}_A^{\delta\geq}$ and $x \in \widetilde{[x]}_B^{\delta\geq}$, $x \in X$ can be otained. Thus, $\underline{\widetilde{OM}}^{\delta\geq}_{A+B}(X) \subseteq X$.

(OU_1) For $\forall x \in X$, owing to the reflexivity of $\widetilde{R}_A^{\delta\geq}$, $\widetilde{[x]}_A^{\delta\geq} \cap X \neq \emptyset$ and $\widetilde{[x]}_B^{\delta\geq} \cap X \neq \emptyset$ hold, then $x \in \overline{\widetilde{OM}}^{\delta\geq}_{A+B}(X)$. Thus, $X \subseteq \overline{\widetilde{OM}}^{\delta\geq}_{A+B}(X)$.

(OL_2) For $\forall x \in \underline{\widetilde{OM}}^{\delta\geq}_{A+B}(\sim X)$, we have $x \in \underline{\widetilde{OM}}^{\delta\geq}_{A+B}(\sim X) \Leftrightarrow \widetilde{[x]}_A^{\delta\geq} \subseteq \sim X$ or $\widetilde{[x]}_B^{\delta\geq} \subseteq \sim X$, then $\widetilde{[x]}_A^{\delta\geq} \subseteq \sim X \quad \widetilde{[x]}_B^{\delta\geq} \subseteq \sim X$,

$$\Leftrightarrow \widetilde{[x]}_A^{\delta\geq} \cap X = \emptyset \quad \widetilde{[x]}_B^{\delta\geq} \cap X = \emptyset,$$

$$\Leftrightarrow x \notin \overline{\widetilde{OM}}^{\delta\geq}_{A+B}(X) \Leftrightarrow x \in \sim \overline{\widetilde{OM}}^{\delta\geq}_{A+B}(X).$$

Thus, equation (OL_2) holds.

(OU_2) We know that $\underline{\widetilde{OM}}^{\delta\geq}_{A+B}(X) = \sim \overline{\widetilde{OM}}^{\delta\geq}_{A+B}(\sim X)$, then $\sim \underline{\widetilde{OM}}^{\delta\geq}_{A+B}(X) = \sim\left(\sim \overline{\widetilde{OM}}^{\delta\geq}_{A+B}(\sim X)\right) = \overline{\widetilde{OM}}^{\delta\geq}_{A+B}(\sim X)$.

(OL_3) In terms of (OL_1), we know that $\underline{\widetilde{OM}}^{\delta\geq}_{A+B}(\emptyset) \subseteq \emptyset$, and it is known that $\emptyset \subseteq \underline{\widetilde{OM}}^{\delta\geq}_{A+B}(\emptyset)$ holds, therefore, $\underline{\widetilde{OM}}^{\delta\geq}_{A+B}(\emptyset) = \emptyset$.

(OU_3) Similar to the proof of (OL_3), we can obtain $\overline{\widetilde{OM}}^{\delta\geq}_{A+B}(\emptyset) = \emptyset$.

(OL_4) $\underline{\widetilde{OM}}^{\delta\geq}_{A+B}(U) = \underline{\widetilde{OM}}^{\delta\geq}_{A+B}(\sim\emptyset)$, according to property (OL_2), $\underline{\widetilde{OM}}^{\delta\geq}_{A+B}(\sim\emptyset) = \sim \overline{\widetilde{OM}}^{\delta\geq}_{A+B}(\emptyset) = \sim\emptyset = U$.

(OU_4) $\overline{\widetilde{OM}}^{\delta\geq}_{A+B}(U) = \overline{\widetilde{OM}}^{\delta\geq}_{A+B}(\sim\emptyset)$,according to property (OU_2), $\overline{\widetilde{OM}}^{\delta\geq}_{A+B}(\sim\emptyset) = \sim \underline{\widetilde{OM}}^{\delta\geq}_{A+B}(\emptyset) = \sim\emptyset = U$.

(OL_5) For $\forall x \in \underline{\widetilde{OM}}^{\delta\geq}_{A+B}(X \cap Y)$, owing to Definition 4.1, we can obtain $\widetilde{[x]}_A^{\delta\geq} \subseteq X \cap Y$ or $\widetilde{[x]}_B^{\delta\geq} \subseteq X \cap Y$, that is, $\widetilde{[x]}_A^{\delta\geq} \subseteq X$ and $\widetilde{[x]}_A^{\delta\geq} \subseteq Y$ holds or $\widetilde{[x]}_B^{\delta\geq} \subseteq X$ and $\widetilde{[x]}_B^{\delta\geq} \subseteq Y$ holds. It is equivalent to $\widetilde{[x]}_A^{\delta\geq} \subseteq X$ or $\widetilde{[x]}_B^{\delta\geq} \subseteq X$ and $\widetilde{[x]}_A^{\delta\geq} \subseteq Y$ or $\widetilde{[x]}_B^{\delta\geq} \subseteq Y$, according to the definition of optimisitc multigranulation lower approximations, then $x \in \underline{\widetilde{OM}}^{\delta\geq}_{A+B}(X) \cap \underline{\widetilde{OM}}^{\delta\geq}_{A+B}(Y)$. Hence, $\underline{\widetilde{OM}}^{\delta\geq}_{A+B}(X \cap Y) \subseteq \underline{\widetilde{OM}}^{\delta\geq}_{A+B}(X) \cap \underline{\widetilde{OM}}^{\delta\geq}_{A+B}(Y)$.

(OU_5)For $\forall x \in \overline{\widetilde{OM}}^{\delta\geq}_{A+B}(X) \cup \overline{\widetilde{OM}}^{\delta\geq}_{A+B}(Y)$, we have $x \in \overline{\widetilde{OM}}^{\delta\geq}_{A+B}(X)$ or $x \in \overline{\widetilde{OM}}^{\delta\geq}_{A+B}(Y)$, that is, $\widetilde{[x]}_A^{\delta\geq} \cap X \neq \emptyset$ and $\widetilde{[x]}_B^{\delta\geq} \cap X \neq \emptyset$ hold or $\widetilde{[x]}_A^{\delta\geq} \cap Y \neq \emptyset$ and $\widetilde{[x]}_B^{\delta\geq} \cap Y \neq \emptyset$ hold. It is equivalent to $\widetilde{[x]}_A^{\delta\geq} \cap (X \cup Y) \neq \emptyset$ and $\widetilde{[x]}_B^{\delta\geq} \cap (X \cup Y) \neq \emptyset$, according to the definition of optimistic multigranulation upper approximations, then

$x \in \overline{\widetilde{OM}_{A+B}^{\delta\geq}}(X \cup Y)$. Hence, $\overline{\widetilde{OM}_{A+B}^{\delta\geq}}(X \cup Y) \supseteq \overline{\widetilde{OM}_{A+B}^{\delta\geq}}(X) \cup \overline{\widetilde{OM}_{A+B}^{\delta\geq}}(Y)$.

(OL_6)Due to $X \subseteq Y$, we can obtain $\underline{\widetilde{OM}_{A+B}^{\delta\geq}}(X \cap Y) = \underline{\widetilde{OM}_{A+B}^{\delta\geq}}(X)$. On account of property (OL_5), it is obvious that $\underline{\widetilde{OM}_{A+B}^{\delta\geq}}(X) \subseteq \underline{\widetilde{OM}_{A+B}^{\delta\geq}}(X) \cap \underline{\widetilde{OM}_{A+B}^{\delta\geq}}(Y)$, then we have $\underline{\widetilde{OM}_{A+B}^{\delta\geq}}(X) = \underline{\widetilde{OM}_{A+B}^{\delta\geq}}(X) \cap \underline{\widetilde{OM}_{A+B}^{\delta\geq}}(Y)$. Therefore, $\underline{\widetilde{OM}_{A+B}^{\delta\geq}}(X) \subseteq \underline{\widetilde{OM}_{A+B}^{\delta\geq}}(Y)$.

(OU_6) Due to $X \subseteq Y$, we can obtain $\overline{\widetilde{OM}_{A+B}^{\delta\geq}}(X \cup Y) = \overline{\widetilde{OM}_{A+B}^{\delta\geq}}(Y)$, according to property (OU_5), it is obvious that $\overline{\widetilde{OM}_{A+B}^{\delta\geq}}(Y) \supseteq \overline{\widetilde{OM}_{A+B}^{\delta\geq}}(X) \cup \overline{\widetilde{OM}_{A+B}^{\delta\geq}}(Y)$, then we have $\overline{\widetilde{OM}_{A+B}^{\delta\geq}}(Y) = \overline{\widetilde{OM}_{A+B}^{\delta\geq}}(X) \cup \overline{\widetilde{OM}_{A+B}^{\delta\geq}}(Y)$. Therefore, $\overline{\widetilde{OM}_{A+B}^{\delta\geq}}(X) \subseteq \overline{\widetilde{OM}_{A+B}^{\delta\geq}}(Y)$.

(OL_7) It is known that $X \subseteq X \cup Y, Y \subseteq X \cup Y$, from property (OL_6), we can learn that $\underline{\widetilde{OM}_{A+B}^{\delta\geq}}(X) \subseteq \underline{\widetilde{OM}_{A+B}^{\delta\geq}}(X \cup Y)$, $\underline{\widetilde{OM}_{A+B}^{\delta\geq}}(Y) \subseteq \underline{\widetilde{OM}_{A+B}^{\delta\geq}}(X \cup Y)$. Therefore, $\underline{\widetilde{OM}_{A+B}^{\delta\geq}}(X \cup Y) \supseteq \underline{\widetilde{OM}_{A+B}^{\delta\geq}}(X) \cup \underline{\widetilde{OM}_{A+B}^{\delta\geq}}(Y)$ holds.

(OU_7) It is obvious that $Y \cap X \subseteq X, X \cap Y \subseteq Y$, from property (OU_6), we can learn that $\overline{\widetilde{OM}_{A+B}^{\delta\geq}}(X) \supseteq \overline{\widetilde{OM}_{A+B}^{\delta\geq}}(X \cap Y)$, $\overline{\widetilde{OM}_{A+B}^{\delta\geq}}(Y) \supseteq \overline{\widetilde{OM}_{A+B}^{\delta\geq}}(X \cap Y)$. There fore, $\overline{\widetilde{OM}_{A+B}^{\delta\geq}}(X \cap Y) \subseteq \overline{\widetilde{OM}_{A+B}^{\delta\geq}}(X) \cap \overline{\widetilde{OM}_{A+B}^{\delta\geq}}(Y)$ holds.

In accordance with the idea of optimistic multigranulation intuitionistic fuzzy rough set, we know that the example described above is a situation of reduced recruitment conditions. Similarly, we can think of the pessimisitc situation as improved recruitment conditions, how to deal with this situation through RST. Then we have the PMNDRS model which we will investigate next.

4.2 Pessimistic Multigranulation Neighborhood Dominance Rough Sets in IFOIS

In this section, the PMNDRS is introduced to handle the approximation problem in IFOIS, and the applications and properties of lower and upper approximation operators are investigated.

Definition 4.2 Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, attributes subsets $A_1, A_2, \ldots, A_n \in AT(n \leq 2^{|AT|})$, the relation $\widetilde{R}_{A_i}^{\delta\geq}$ represents neighborhood dominance relations on attribute subsets $A_i(i = 1, 2, \ldots, n)$. $\forall X \in P(U)$, $\widetilde{[x]}_{A_i}^{\delta\geq} = \{y \mid (x, y) \in \widetilde{R}_{A_i}^{\delta\geq}\}$ is called the i-th neighborhood dominance class which contains x with respect to the i-th neighborhood dominance relation $\widetilde{R}_{A_i}^{\delta\geq}$. The pessimistic multigranulation lower and upper approximations of X are defined as follows:

$$\underline{\widetilde{PM}_{\sum_{i=1}^{n} A_i}^{\delta\geq}}(X) = \left\{ x \in U \,\middle|\, \bigwedge_{i=1}^{n} (\widetilde{[x]}_{A_i}^{\delta\geq} \subseteq X) \right\},$$

$$\overline{\widetilde{PM}_{\sum_{i=1}^{n} A_i}^{\delta\geq}}(X) = \left\{ x \in U \,\middle|\, \bigvee_{i=1}^{n} (\widetilde{[x]}_{A_i}^{\delta\geq} \cap X \neq \emptyset) \right\},$$

where logical operations "$\wedge$" represents "and", "$\vee$" represents "or". $\underline{\widetilde{PM}_{\sum_{i=1}^{n} A_i}^{\delta\geq}}(X)$ and $\overline{\widetilde{PM}_{\sum_{i=1}^{n} A_i}^{\delta\geq}}(X)$ are called pessimistic multigranulation lower and upper approximation operators. If $\underline{\widetilde{PM}_{\sum_{i=1}^{n} A_i}^{\delta\geq}}(X) \neq \overline{\widetilde{PM}_{\sum_{i=1}^{n} A_i}^{\delta\geq}}(X)$, we call X a rough set with respect to neighborhood dominance relations $\widetilde{R}_{A_i}^{\delta\geq}$ $(i = 1, 2, \ldots, n)$, otherwise, we call X a definable set. The three disjoint regions of the target set X are respectively $POS(X) = \underline{\widetilde{PM}_{\sum_{i=1}^{n} A_i}^{\delta\geq}}(X), NEG(X) = \sim \overline{\widetilde{PM}_{\sum_{i=1}^{n} A_i}^{\delta\geq}}(X)$ and $BND(X) = \overline{\widetilde{PM}_{\sum_{i=1}^{n} A_i}^{\delta\geq}}(X) - \underline{\widetilde{PM}_{\sum_{i=1}^{n} A_i}^{\delta\geq}}(X)$.

Example 4.2 (Continued from Example 4.1) We have discussed the recruitment in the optimistic situation, then we consider the application of rough set model in recruitment under pessimistic situations. In the pessimistic situation, the difficulty of employment is improved. It is required that interviewers who meet all conditions will be hired, so we have raised two questions.

Question 3 If the company requires the candidate to meet all the conditions, who will be surely hired.
Question 4 If the company requires the candidate to meet at least one of conditions, who will be possibly hired.

According to Definition 4.2, we can obtain

$$\underline{\widetilde{PM}_{1+2}^{\delta\geq}}(X) = \{x_4, x_5, x_6, x_9, x_{10}\}; \qquad \underline{\widetilde{PM}_{1+2}^{\delta\geq}}(X) = \underline{\widetilde{R}_1^{\delta\geq}}(X) \cap \underline{\widetilde{R}_2^{\delta\geq}}(X);$$

$$\overline{\widetilde{PM}_{1+2}^{\delta\geq}}(X) = \{x_1, x_4, x_5, x_6, x_9, x_{10}\}. \qquad \overline{\widetilde{PM}_{1+2}^{\delta\geq}}(X) = \overline{\widetilde{R}_1^{\delta\geq}}(X) \cup \overline{\widetilde{R}_2^{\delta\geq}}(X).$$

From the lower and upper approximations, x_5, x_6, x_{10} will surely be employed when both conditions are considered. Meanwhile, $x_3, x_5, x_6, x_8, x_9, x_{10}$ will possibly be employed when the company considered at least one condition. Futhermore, the properties of PMNDRS can be obtained.

Proposition 4.3 *Let* $\widetilde{I}^{\geq} = (U, AT, F)$ *be an IFOIS,* $A_i \subseteq AT(i = 1, 2, \ldots, n, n \leq 2^{|AT|})$, $0 \leq \delta_1 \leq \delta_2 \leq 1$, *then*

(1) $\underline{\widetilde{PM}_{\sum_{i=1}^{n} A_i}^{\delta_2\geq}}(X) \subseteq \underline{\widetilde{PM}_{\sum_{i=1}^{n} A_i}^{\delta_1\geq}}(X)$,

(2) $\overline{\widetilde{PM}}^{\delta_1\geq}_{\sum_{i=1}^{n}A_i}(X) \subseteq \overline{\widetilde{PM}}^{\delta_2\geq}_{\sum_{i=1}^{n}A_i}(X)$.

Proposition 4.4 *Let* $\widetilde{I}^{\geq}=(U,AT,F)$ *be an IFOIS,* $A_i \subseteq AT(i=1,2,\ldots,n,n\leq 2^{|AT|})$. *The pessimistic multigranulation lower and upper approximations under IFOIS have following properties*

$(PL_1)\underline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(X) \subseteq X;$ (Contraction)

$(PU_1)\overline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(X) \supseteq X;$ (Extension)

$(PL_2)\underline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(\sim X) = \sim \overline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(X);$ (Duality)

$(PU_2)\overline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(\sim X) = \sim \underline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(X);$ (Duality)

$(PL_3)\underline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(\emptyset) = \emptyset;$ (Normality)

$(PU_3)\overline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(\emptyset) = \emptyset;$ (Normality)

$(PL_4)\underline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(U) = U;$ (Co-normality)

$(PU_4)\overline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(U) = U;$ (Co-normality)

$(PL_5)\underline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(X\cap Y) = \underline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(X) \cap \underline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(Y);$ (L-multiplication)

$(PU_5)\overline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(X\cup Y) = \overline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(X) \cup \overline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(Y);$ (L-addition)

$(PL_6)X \subseteq Y \Rightarrow \underline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(X) \subseteq \underline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(Y);$ (Granularity)

$(PU_6)X \subseteq Y \Rightarrow \overline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(X) \subseteq \overline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(Y);$ (Granularity)

$(PL_7)\underline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(X\cup Y) \supseteq \underline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(X) \cup \underline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(Y);$ (U-addition)

$(PU_7)\overline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(X\cap Y) \subseteq \overline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(X) \cap \overline{\widetilde{PM}}^{\delta\geq}_{\sum_{i=1}^{n}A_i}(Y).$ (U-multiplication)

Proof For a better explanation, we will give the proof of the properties based on two neighborhood dominance relations. Suppose $n=2$, $A,B\subseteq AT$, when $A=B$, it is obviously true. Therefore, when $A\neq B$, then

(PL_1)For $\forall x \in \underline{\widetilde{PM}}^{\delta\geq}_{A+B}(X)$, according to Definition 4.2, we have $\widetilde{[x]}^{\delta\geq}_{A} \subseteq X$ and $\widetilde{[x]}^{\delta\geq}_{B} \subseteq X$. Because the neighborhood dominance relation $\widetilde{R}^{\delta\geq}_{A}$ satisfies reflexivity, then $x\in\widetilde{[x]}^{\delta\geq}_{A}$ and $x\in\widetilde{[x]}^{\delta\geq}_{B}$, we can obtain $x\in X$. Thus, $\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(X)\subseteq X$.

(PU_1) For $\forall x\in X$, due to the reflexivity of $\widetilde{R}^{\delta\geq}_{A}$, we have $\widetilde{[x]}^{\delta\geq}_{A}\cap X\neq\emptyset$ and $\widetilde{[x]}^{\delta\geq}_{B}\cap X\neq\emptyset$, then $x\in\overline{\widetilde{PM}}^{\delta\geq}_{A+B}(X)$. Thus, $X\subseteq\overline{\widetilde{PM}}^{\delta\geq}_{A+B}(X)$.

(PL_2) For $\forall x\in\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(\sim X)$, we have $x\in\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(\sim X)\Leftrightarrow\widetilde{[x]}^{\delta\geq}_{A}\subseteq\sim X$ *and* $\widetilde{[x]}^{\delta\geq}_{B}\subseteq\sim X$, then

$\widetilde{[x]}^{\delta\geq}_{A}\subseteq\sim X \quad \widetilde{[x]}^{\delta\geq}_{B}\subseteq\sim X,$

$\Leftrightarrow\widetilde{[x]}^{\delta\geq}_{A}\cap X=\emptyset \quad \widetilde{[x]}^{\delta\geq}_{B}\cap X=\emptyset,$

$\Leftrightarrow x\notin\overline{\widetilde{PM}}^{\delta\geq}_{A+B}(X)\Leftrightarrow x\in\sim\overline{\widetilde{PM}}^{\delta\geq}_{A+B}(X).$

(PU_2) We know that $\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(X)=\sim\overline{\widetilde{PM}}^{\delta\geq}_{A+B}(\sim X)$, then $\sim\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(X)=\sim\left(\sim\overline{\widetilde{PM}}^{\delta\geq}_{A+B}(\sim X)\right)=\overline{\widetilde{PM}}^{\delta\geq}_{A+B}(\sim X)$.

(PL_3) In terms of (PL_1), we know that $\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(\emptyset)\subseteq\emptyset$, and it is known that $\emptyset\subseteq\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(\emptyset)$ holds, therefore, $\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(\emptyset)=\emptyset$.

(PU_3) Similar to the proof of (PL_3), we can obtain $\overline{\widetilde{PM}}^{\delta\geq}_{A+B}(\emptyset)=\emptyset$.

(PL_4) $\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(U)=\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(\sim\emptyset)$, according to property (PL_2), $\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(\sim\emptyset)=\sim\overline{\widetilde{PM}}^{\delta\geq}_{A+B}(\emptyset)=\sim\emptyset=U$.

(PU_4) $\overline{\widetilde{PM}}^{\delta\geq}_{A+B}(U)=\overline{\widetilde{PM}}^{\delta\geq}_{A+B}(\sim\emptyset)$, according to property (PU_2), $\overline{\widetilde{PM}}^{\delta\geq}_{A+B}(\sim\emptyset)=\sim\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(\emptyset)=\sim\emptyset=U$.

(PL_5) For $\forall x\in\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(X\cap Y)$, thanks to Definition 4.2, we can obtain $\forall x\in\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(X\cap Y)\Leftrightarrow\widetilde{[x]}^{\delta\geq}_{A}\subseteq X\cap Y$ and $\widetilde{[x]}^{\delta\geq}_{B}\subseteq X\cap Y$, that is, $\widetilde{[x]}^{\delta\geq}_{A}\subseteq X,\widetilde{[x]}^{\delta\geq}_{A}\subseteq Y$, hold and $\widetilde{[x]}^{\delta\geq}_{B}\subseteq X,\widetilde{[x]}^{\delta\geq}_{B}\subseteq Y$ hold. It is equivalent to $\widetilde{[x]}^{\delta\geq}_{A}\subseteq X,\widetilde{[x]}^{\delta\geq}_{B}\subseteq X$ and $\widetilde{[x]}^{\delta\geq}_{A}\subseteq Y,\widetilde{[x]}^{\delta\geq}_{B}\subseteq Y$, according to the definition of the pessimistic multigranulation lower approximations, then $x\in\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(X)\cap\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(Y)$. Hence, $\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(X\cap Y)=\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(X)\cap\underline{\widetilde{PM}}^{\delta\geq}_{A+B}(Y)$.

(PU_5) For $\forall x\in\overline{\widetilde{PM}}^{\delta\geq}_{A+B}(X)\cup\overline{\widetilde{PM}}^{\delta\geq}_{A+B}(Y)$, we have $x\in\overline{\widetilde{PM}}^{\delta\geq}_{A+B}(X)$ or $x\in\overline{\widetilde{PM}}^{\delta\geq}_{A+B}(Y)$, that is, $\widetilde{[x]}^{\delta\geq}_{A}\cap X\neq\emptyset$ or $\widetilde{[x]}^{\delta\geq}_{B}\cap X\neq\emptyset$ holds or $\widetilde{[x]}^{\delta\geq}_{A}\cap Y\neq\emptyset$ or $\widetilde{[x]}^{\delta\geq}_{B}\cap Y\neq\emptyset$ holds. It is equivalent to $\widetilde{[x]}^{\delta\geq}_{A}\cap(X\cup Y)\neq\emptyset$ or

$\widetilde{[x]}^{\delta\geq}_B \cap (X\cup Y) \neq \emptyset$, according to the definition of pessimistic multigranulation upper approximations, then $x \in \overline{\widetilde{PM}^{\delta\geq}_{A+B}}(X\cup Y)$. Hence, $\overline{\widetilde{PM}^{\delta\geq}_{A+B}}(X\cup Y) = \overline{\widetilde{PM}^{\delta\geq}_{A+B}}(X) \cup \overline{\widetilde{PM}^{\delta\geq}_{A+B}}(Y)$.

(PL_6) Owing to $X \subseteq Y$, $\underline{\widetilde{PM}^{\delta\geq}_{A+B}}(X\cap Y) = \underline{\widetilde{PM}^{\delta\geq}_{A+B}}(X)$ can be obtained. Owing to property (PL_5), it is obvious that $\underline{\widetilde{PM}^{\delta\geq}_{A+B}}(X\cap Y) = \underline{\widetilde{PM}^{\delta\geq}_{A+B}}(X) \cap \underline{\widetilde{PM}^{\delta\geq}_{A+B}}(Y)$, then $\underline{\widetilde{PM}^{\delta\geq}_{A+B}}(X) = \underline{\widetilde{PM}^{\delta\geq}_{A+B}}(X) \cap \underline{\widetilde{PM}^{\delta\geq}_{A+B}}(Y)$. Thus, $\underline{\widetilde{PM}^{\delta\geq}_{A+B}}(X) \subseteq \underline{\widetilde{PM}^{\delta\geq}_{A+B}}(Y)$.

(PU_6) Owing to $X \subseteq Y$, we can obtain $\overline{\widetilde{PM}^{\delta\geq}_{A+B}}(X\cup Y) = \overline{\widetilde{PM}^{\delta\geq}_{A+B}}(Y)$. According to property (PU_5), it is obvious that $\overline{\widetilde{PM}^{\delta\geq}_{A+B}}(X\cup Y) = \overline{\widetilde{PM}^{\delta\geq}_{A+B}}(X) \cup \overline{\widetilde{PM}^{\delta\geq}_{A+B}}(Y)$, then $\overline{\widetilde{PM}^{\delta\geq}_{A+B}}(Y) = \overline{\widetilde{PM}^{\delta\geq}_{A+B}}(X) \cup \overline{\widetilde{PM}^{\delta\geq}_{A+B}}(Y)$. Therefore, $\overline{\widetilde{PM}^{\delta\geq}_{A+B}}(X) \subseteq \overline{\widetilde{PM}^{\delta\geq}_{A+B}}(Y)$.

(PL_7) It is known that $X \subseteq X\cup Y, Y \subseteq X \cup Y$, from property (PL_6), we can learn that $\underline{\widetilde{PM}^{\delta\geq}_{A+B}}(X) \subseteq \underline{\widetilde{PM}^{\delta\geq}_{A+B}}(X\cup Y)$, $\underline{\widetilde{PM}^{\delta\geq}_{A+B}}(Y) \subseteq \underline{\widetilde{PM}^{\delta\geq}_{A+B}}(X\cup Y)$. Therefore, $\underline{\widetilde{PM}^{\delta\geq}_{A+B}}(X) \cup \underline{\widetilde{PM}^{\delta\geq}_{A+B}}(Y) \subseteq \underline{\widetilde{PM}^{\delta\geq}_{A+B}}(X\cup Y)$.

(PU_7) It is obvious that $Y \cap X \subseteq X, X\cap Y \subseteq Y$, from property (PU_6), we can learn that $\overline{\widetilde{PM}^{\delta\geq}_{A+B}}(X) \supseteq \overline{\widetilde{PM}^{\delta\geq}_{A+B}}(X\cap Y)$, $\overline{\widetilde{PM}^{\delta\geq}_{A+B}}(Y) \supseteq \overline{\widetilde{PM}^{\delta\geq}_{A+B}}(X\cap Y)$. Therefore, $\overline{\widetilde{PM}^{\delta\geq}_{A+B}}(X\cap Y) \subseteq \overline{\widetilde{PM}^{\delta\geq}_{A+B}}(X) \cap \overline{\widetilde{PM}^{\delta\geq}_{A+B}}(Y)$.

5 The Uncertainty Measures of MNDRS in IFOIS

According to Pawlak rough set, the uncertainty exists because of the boundary region of the rough set. The larger the boundary region of the rough set, the uncertainty of the rough set becomes greater. Similarly, there exists uncertainty in MNDRS models. This section will mainly introduce two elementary measures to measure the uncertainty of MNDRS in IFOIS. Furthermore, the relationship between single-granularity NDRS and MNDRS will be explored in this section.

Proposition 5.1 *Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, and $A_i \subseteq AT (i = 1, 2, \ldots, n, n \leq 2^{|AT|})$. For $\forall X \subseteq U$, there have*

(1) $\underline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^n A_i}}(X) = \bigcup_{i=1}^n \underline{\widetilde{R}^{\delta\geq}_{A_i}}(X)$, $\overline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^n A_i}}(X) = \bigcap_{i=1}^n \overline{\widetilde{R}^{\delta\geq}_{A_i}}(X)$.

(2) $\underline{\widetilde{PM}^{\delta\geq}_{\sum_{i=1}^n A_i}}(X) = \bigcap_{i=1}^n \underline{\widetilde{R}^{\delta\geq}_{A_i}}(X)$, $\overline{\widetilde{PM}^{\delta\geq}_{\sum_{i=1}^n A_i}}(X) = \bigcup_{i=1}^n \overline{\widetilde{R}^{\delta\geq}_{A_i}}(X)$.

Proof Since the number of granulations in IFOIS is finite, we only need prove these properties in IFOIS which has two neighborhood dominance relations $(A, B \subseteq AT)$ for convenience.

(1) For $\forall x \in \underline{\widetilde{OM}^{\delta\geq}_{A+B}}(X)$, we have $\widetilde{[x]}^{\delta\geq}_A \subseteq X$ or $\widetilde{[x]}^{\delta\geq}_B \subseteq X$. We can obtain $x \in \underline{\widetilde{R}^{\delta\geq}_A}(X)$ or $x \in \underline{\widetilde{R}^{\delta\geq}_B}(X) \Leftrightarrow x \in \underline{\widetilde{R}^{\delta\geq}_A}(X) \cup \underline{\widetilde{R}^{\delta\geq}_B}(X)$. Hence, $\underline{\widetilde{OM}^{\delta\geq}_{A+B}}(X) = \underline{\widetilde{R}^{\delta\geq}_A}(X) \cup \underline{\widetilde{R}^{\delta\geq}_B}(X)$.

For $\forall x \in \overline{\widetilde{OM}^{\delta\geq}_{A+B}}(X)$, we have $\widetilde{[x]}^{\delta\geq}_A \cap X \neq \emptyset$ and $\widetilde{[x]}^{\delta\geq}_B \cap X \neq \emptyset$. We can obtain $x \in \overline{\widetilde{R}^{\delta\geq}_A}(X)$ and $x \in \overline{\widetilde{R}^{\delta\geq}_B}(X) \Leftrightarrow x \in \overline{\widetilde{R}^{\delta\geq}_A}(X) \cap \overline{\widetilde{R}^{\delta\geq}_B}(X)$. Hence, $\overline{\widetilde{OM}^{\delta\geq}_{A+B}}(X) = \overline{\widetilde{R}^{\delta\geq}_A}(X) \cap \overline{\widetilde{R}^{\delta\geq}_B}(X)$.

(2)For $\forall x \in \underline{\widetilde{PM}^{\delta\geq}_{A+B}}(X)$, we have $\widetilde{[x]}^{\delta\geq}_A \subseteq X$ and $\widetilde{[x]}^{\delta\geq}_B \subseteq X \Leftrightarrow x \in \underline{\widetilde{R}^{\delta\geq}_A}(X)$ and $x \in \underline{\widetilde{R}^{\delta\geq}_B}(X) \Leftrightarrow x \in \underline{\widetilde{R}^{\delta\geq}_A}(X) \cap \underline{\widetilde{R}^{\delta\geq}_B}(X)$. Hence, $\underline{\widetilde{PM}^{\delta\geq}_{A+B}}(X) = \underline{\widetilde{R}^{\delta\geq}_A}(X) \cap \underline{\widetilde{R}^{\delta\geq}_B}(X)$.

For $\forall x \in \overline{\widetilde{PM}^{\delta\geq}_{A+B}}(X)$, we have $\widetilde{[x]}^{\delta\geq}_A \cap X \neq \emptyset$ or $\widetilde{[x]}^{\delta\geq}_B \cap X \neq \emptyset \Leftrightarrow x \in \overline{\widetilde{R}^{\delta\geq}_A}(X)$ or $x \in \overline{\widetilde{R}^{\delta\geq}_B}(X) \Leftrightarrow x \in \overline{\widetilde{R}^{\delta\geq}_A}(X) \cup \overline{\widetilde{R}^{\delta\geq}_B}(X)$. Hence, $\overline{\widetilde{OM}^{\delta\geq}_{A+B}}(X) = \overline{\widetilde{R}^{\delta\geq}_A}(X) \cup \overline{\widetilde{R}^{\delta\geq}_B}(X)$.

The proof is complete. □

Proposition 5.2 *Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, and $A_i \subseteq AT (i = 1, 2, \ldots, n, n \leq 2^{|AT|})$. For $\forall X, Y \subseteq U$, then*

(1) $\underline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^n A_i}}(X\cap Y) = \bigcup_{i=1}^n \left(\underline{\widetilde{R}^{\delta\geq}_{A_i}}(X) \cap \underline{\widetilde{R}^{\delta\geq}_{A_i}}(Y)\right)$,
$\underline{\widetilde{PM}^{\delta\geq}_{\sum_{i=1}^n A_i}}(X\cap Y) = \bigcap_{i=1}^n \left(\underline{\widetilde{R}^{\delta\geq}_{A_i}}(X) \cap \underline{\widetilde{R}^{\delta\geq}_{A_i}}(Y)\right)$.

(2) $\overline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^n A_i}}(X\cap Y) = \bigcap_{i=1}^n \left(\overline{\widetilde{R}^{\delta\geq}_{A_i}}(X) \cap \overline{\widetilde{R}^{\delta\geq}_{A_i}}(Y)\right)$,
$\overline{\widetilde{PM}^{\delta\geq}_{\sum_{i=1}^n A_i}}(X\cap Y) = \bigcup_{i=1}^n \left(\overline{\widetilde{R}^{\delta\geq}_{A_i}}(X) \cup \overline{\widetilde{R}^{\delta\geq}_{A_i}}(Y)\right)$.

The proof can be obtained by Proposition 5.1.

Proposition 5.3 *Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, and $A_i \subseteq AT (i = 1, 2, \ldots, n, n \leq 2^{|AT|})$. For $\forall X \subseteq U$, then*

(1) $\underline{\widetilde{PM}^{\delta\geq}_{\sum_{i=1}^n A_i}}(X) \subseteq \underline{\widetilde{R}^{\delta\geq}_{A_i}}(X) \subseteq \underline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^n A_i}}(X) \subseteq \underline{\widetilde{R}^{\delta\geq}_{\cup_{i=1}^n}}(X)$.

(2) $\overline{\widetilde{PM}^{\delta\geq}_{\sum_{i=1}^{n}A_i}}(X) \supseteq \overline{\widetilde{R}^{\delta\geq}_{A_i}}(X) \supseteq \overline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^{n}A_i}}(X) \supseteq \overline{\widetilde{R}^{\delta\geq}_{\cup_{i=1}^{n}}}(X)$.

The proof can be obtained by Proposition 5.1, Definitions 4.1 and 4.2.

From above the definition of single- granularity NDRS in IFOIS, we have defined two uncertainty measures of single- granularity NDRS. In the following, we will discuss the roughness and dependence degree of MNDRS in IFOIS, which is similar to single-granularity NDRS.

Definition 5.1 Let $\widetilde{I}^{\geq}=(U,AT,F)$ be an IFOIS, and $A_i \subseteq AT(i=1,2,\ldots,n,n\leq 2^{|AT|})$. For $\forall X\subseteq U$, the optimistic and pessimistic rough measures of X can be defined as

$$\rho_O^{\delta\geq}(X,\textstyle\sum_{i=1}^{n}A_i)=1-\dfrac{\left|\underline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^{n}A_i}}(X)\right|}{\left|\overline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^{n}A_i}}(X)\right|},$$

$$\rho_P^{\delta\geq}(X,\textstyle\sum_{i=1}^{n}A_i)=1-\dfrac{\left|\underline{\widetilde{PM}^{\delta\geq}_{\sum_{i=1}^{n}A_i}}(X)\right|}{\left|\overline{\widetilde{PM}^{\delta\geq}_{\sum_{i=1}^{n}A_i}}(X)\right|},$$

where $X\neq\emptyset$. From the definition we can obtain that if $\overline{\widetilde{OM}^{\delta\geq}_{\sum_{i=1}^{n}A_i}}(X)=\emptyset$ or $\overline{\widetilde{PM}^{\delta\geq}_{\sum_{i=1}^{n}A_i}}(X)=\emptyset$, then $\rho_O^{\delta\geq}(X,\sum_{i=1}^{n}A_i)=1$ or $\rho_P^{\delta\geq}(X,\sum_{i=1}^{n}A_i)=1$. The accuracy measures of X are $\alpha_O^{\delta\geq}(X,\sum_{i=1}^{n}A_i)=1-\rho_O^{\delta\geq}(X,\sum_{i=1}^{n}A_i)$, $\alpha_P^{\delta\geq}(X,\sum_{i=1}^{n}A_i)=1-\rho_P^{\delta\geq}(X,\sum_{i=1}^{n}A_i)$.

Algorithm 1: An algorithm for computing the roughness and the dependence degree of NDRS in IFODIS

Input : An IFODIS $\widetilde{I}^{\geq}=(U,AT\cup\{d\},F)$, where attribute sets $A_i\subseteq AT(i=1,2,...,n)$
Output : The roughness and the degree of dependence of NRS and NDRS in IFODIS

begin
  Compute $U/d=\{D_1,D_2,\cdots,D_m\}$; $X=\widetilde{[x^*]}^{\delta\geq}_{AT}-\{x^*\}$
  for $i=1:n$; **do**
    for $\forall x\in U$ **do**
      compute $\widetilde{[x]}^{\delta\geq}_{A_i}$; $\widetilde{[x]}^{\delta\geq}_{A_i}$
    end
    for $j=1:m$ **do**
      let $\underline{\widetilde{R}^{\delta\geq}_{A_i}}(D_j)=\emptyset$, $\overline{\widetilde{R}^{\delta\geq}_{A_i}}(D_j)=\emptyset$; $\underline{\widetilde{R}^{\delta}_{A_i}}(D_j)=\emptyset$, $\overline{\widetilde{R}^{\delta}_{A_i}}(D_j)=\emptyset$
      $\underline{\widetilde{R}^{\delta\geq}_{A_i}}(X)=\emptyset$, $\overline{\widetilde{R}^{\delta\geq}_{A_i}}(X)=\emptyset$; $\underline{\widetilde{R}^{\delta}_{A_i}}(X)=\emptyset$, $\overline{\widetilde{R}^{\delta}_{A_i}}(X)=\emptyset$
    end
    for $j=1:m$ **do**
      for $\forall x\in U$ **do**
        if $\widetilde{[x]}^{\delta\geq}_{A_i}\subseteq D_j$ if $\widetilde{[x]}^{\delta}_{A_i}\subseteq X$ then
        $\underline{\widetilde{R}^{\delta\geq}_{A_i}}(D_j)=\underline{\widetilde{R}^{\delta\geq}_{A_i}}(D_j)\cup\{x\}$ $\underline{\widetilde{R}^{\delta}_{A_i}}(X)=\underline{\widetilde{R}^{\delta}_{A_i}}(X)\cup\{x\}$
        if $x\in D_j$ then   if $x\in X$ then
        $\overline{\widetilde{R}^{\delta\geq}_{A_i}}(D_j)=\overline{\widetilde{R}^{\delta\geq}_{A_i}}(D_j)\cup\widetilde{[x]}^{\delta\geq}_{A}$ $\overline{\widetilde{R}^{\delta}_{A_i}}(X)=\overline{\widetilde{R}^{\delta}_{A_i}}(X)\cup\widetilde{[x]}^{\delta}_{A}$
      end
      $\gamma(A_i,D)=\emptyset$
      for $i=1:m$ **do**
        $\rho(\widetilde{R}^{\delta\geq}_{A_i},X)=1-\dfrac{|\underline{\widetilde{R}^{\delta\geq}_{A_i}}(X)|}{|\overline{\widetilde{R}^{\delta\geq}_{A_i}}(X)|}$; $\rho(\widetilde{R}^{\delta}_{A_i},X)=1-\dfrac{|\underline{\widetilde{R}^{\delta}_{A_i}}(X)|}{|\overline{\widetilde{R}^{\delta}_{A_i}}(X)|}$
        $\gamma(A_i,D)=\gamma(A_i,D)+\dfrac{|\underline{\widetilde{R}^{\delta\geq}_{A_i}}(D_j)|}{|U|}$; $\gamma^{\delta}(A_j,D)=\gamma^{\delta}(A_i,D)+\dfrac{|\underline{\widetilde{R}^{\delta}_{A_i}}(D_j)|}{|U|}$
      end
    end
    return : $\rho(\widetilde{R}^{\delta\geq}_{A_i},X)$, $\gamma(A_i,D)$;$\rho(\widetilde{R}^{\delta}_{A_i},X)$, $\gamma^{\delta}(A_i,D)=\gamma^{\delta}(A_i,D)$.
  end
end

Algorithm 2: The algorithm for computing the roughness and the dependence degree of MNDRS in IFODIS

Input : An IFODIS $\widetilde{I}^{\geq}=(U,AT\cup\{d\},F)$, where attribute sets $A_i\subseteq AT(i=1,2,...,n)$, $X=\widetilde{[x^*]}_{A_i}^{\delta\geq}-\{x^*\}$;
Decision classes $U/d=\{D_1,D_2,\cdots,D_m\}$, the approximation sets $\underline{\widetilde{R}}_{A_i}^{\delta\geq}(D_j)$, $\overline{\widetilde{R}}_{A_i}^{\delta\geq}(D_j)$.

Output : The roughness and the degree of dependence of MNDRS
$\rho_O^{\delta\geq}(X,\sum_{i=1}^n A_i)$, $\rho_P^{\delta\geq}(X,\sum_{i=1}^n A_i)$ and $\gamma_{OM}^{\delta\geq}(d,\sum_{i=1}^n A_i)$, $\gamma_{PM}^{\delta\geq}(d,\sum_{i=1}^n A_i)$.

begin
  for $j=1:m$ **do**
    $\underline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(D_j)=\emptyset$, $\overline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(D_j)=U$; $\underline{\widetilde{PM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(D_j)=U$, $\overline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(D_j)=\emptyset$
    $\underline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(X)=\emptyset$, $\overline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(X)=U$; $\underline{\widetilde{PM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(X)=U$, $\overline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(X)=\emptyset$.
    for $i=1:n$ **do**
      $\underline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(D_j)=\underline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(D_j)\cup\underline{\widetilde{R}}_{A_i}^{\delta\geq}(D_j)$, $\overline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(D_j)=\overline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(D_j)\cap\overline{\widetilde{R}}_{A_i}^{\delta\geq}(D_j)$;
      $\underline{\widetilde{PM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(D_j)=\underline{\widetilde{PM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(D_j)\cap\underline{\widetilde{R}}_{A_i}^{\delta\geq}(D_j)$, $\overline{\widetilde{PM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(D_j)=\overline{\widetilde{PM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(D_j)\cup\overline{\widetilde{R}}_{A_i}^{\delta\geq}(D_j)$;
      $\underline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(X)=\underline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(X)\cup\underline{\widetilde{R}}_{A_i}^{\delta\geq}(X)$, $\overline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(X)=\overline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(X)\cap\overline{\widetilde{R}}_{A_i}^{\delta\geq}(X)$;
      $\underline{\widetilde{PM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(X)=\underline{\widetilde{PM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(X)\cap\underline{\widetilde{R}}_{A_i}^{\delta\geq}(X)$, $\overline{\widetilde{PM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(X)=\overline{\widetilde{PM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(X)\cup\overline{\widetilde{R}}_{A_i}^{\delta\geq}(X)$.
    end
  end
  for $j=1:m$ **do**
    $\rho_O^{\delta\geq}(X,\sum_{i=1}^n A_i)=1-\dfrac{\left|\underline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(X)\right|}{\left|\overline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(X)\right|}$;
    $\rho_P^{\delta\geq}(X,\sum_{i=1}^n A_i)=1-\dfrac{\left|\underline{\widetilde{PM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(X)\right|}{\left|\overline{\widetilde{PM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(X)\right|}$.
  end
  $\gamma_{OM}^{\delta\geq}(D_j,\sum_{i=1}^n A_i)=\emptyset$;
  $\gamma_{PM}^{\delta\geq}(D_j,\sum_{i=1}^n A_i)=\emptyset$.
  for $j=1:m$ **do**
    $\gamma_{OM}^{\delta\geq}(D_j,\sum_{i=1}^n A_i)=\gamma_{OM}^{\delta\geq}(D_j,\sum_{i=1}^n A_i)+\dfrac{\left|\underline{\widetilde{R}}_{A_i}^{\delta\geq}(D_j)\right|}{|U|}$;
    $\gamma_{PM}^{\delta\geq}(D_j,\sum_{i=1}^n A_i)=\gamma_{PM}^{\delta\geq}(D_j,\sum_{i=1}^n A_i)+\dfrac{\left|\underline{\widetilde{R}}_{A_i}^{\delta\geq}(D_j)\right|}{|U|}$
  end
  return:$\rho_O^{\delta\geq}(X,\sum_{i=1}^n A_i)$, $\rho_P^{\delta\geq}(X,\sum_{i=1}^n A_i)$;
  $\gamma_{OM}^{\delta\geq}(D_j,\sum_{i=1}^n A_i)$, $\gamma_{PM}^{\delta\geq}(D_j,\sum_{i=1}^n A_i)$.
end

Proposition 5.4 Let $\widetilde{I}^{\geq}=(U,AT,F)$ be an IFOIS, and $A_i\subseteq AT(i=1,2,\ldots,n,n\leq 2^{|AT|})$. For $\forall X\subseteq U$, then $\rho^{\delta\geq}_{\bigcup_{i=1}^n A_i}(X)\leq\rho_O^{\delta\geq}(X,\sum_{i=1}^n A_i)\leq\rho_{A_i}^{\delta\geq}(X)\leq\rho_P^{\delta\geq}(X,\sum_{i=1}^n A_i)$.

Definition 5.2 Let $\widetilde{I}^{\geq}=(U,AT,F)$ be an IFOIS, and $A_i\subseteq AT(i=1,2,\ldots,n,n\leq 2^{|AT|})$. The approximate quality of decision attribute d by $\sum_{i=1}^n A_i$ called the optimistic and pessimistic degree of dependence is defined as

$$\gamma_{OM}^{\delta\geq}(D,\sum_{i=1}^n A_i)=\frac{1}{|U|}\left(\sum_{t=1}^{s}\left|\underline{\widetilde{OM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(D_t)\right|\right),$$

$$\gamma_{PM}^{\delta\geq}(D,\sum_{i=1}^n A_i)=\frac{1}{|U|}\left(\sum_{t=1}^{s}\left|\underline{\widetilde{PM}}_{\sum_{i=1}^n A_i}^{\delta\geq}(D_t)\right|\right).$$

Table 4 The time complexity of Algorithms 1 and 2

Steps	NRS	NDRS	MNDRS
1–10	$O(\lvert AT\rvert\lvert U\rvert)+O(\lvert AT\rvert\lvert D_j\rvert)$	$O(\lvert AT\rvert\lvert U\rvert)+O(\lvert AT\rvert\lvert D_j\rvert)$	$O(\lvert AT\rvert\lvert D_j\rvert)$
11–15	$O(\lvert U\rvert\lvert D_j\rvert)$	$O(\lvert U\rvert\lvert D_j\rvert)$	$O(\lvert D_j\rvert)$
18–22	$O(\lvert AT\rvert\lvert D_j\rvert)$	$O(\lvert AT\rvert\lvert D_j\rvert)$	$O(\lvert D_j\rvert)$
Total	$O(\lvert AT\rvert\lvert U\rvert+2\lvert AT\rvert\lvert D_j\rvert+\lvert U\rvert\lvert D_j\rvert)$	$O(\lvert AT\rvert\lvert U\rvert+2\lvert AT\rvert\lvert D_j\rvert+\lvert U\rvert\lvert D_j\rvert)$	$O(\lvert AT\rvert\lvert D_j\rvert+2\lvert D_j\rvert)$

Proposition 5.5 *Let* $\widetilde{I}^{\geq}=(U,AT,F)$ *be an IFOIS, and* $A_i\subseteq AT(i=1,2,\ldots,n,n\leq 2^{|AT|})$. *For* $\forall X\subseteq U$, *then* $\gamma_{PM}^{\delta\geq}(D,\sum_{i=1}^{n}A_i)\leq\gamma^{\delta\geq}(D,A_i)\leq\gamma_{OM}^{\delta\geq}(D,\sum_{i=1}^{n}A_i)\leq\gamma^{\delta\geq}(D,\bigcup_{i=1}^{n}A_i)$.

6 Algorithms for Computing the Roughness and the Degree of Dependence in IFODIS

In this section, we design two algorithms based on NDRS, MNDRS, and other compared models to verify the feasibility of proposed methods through uncertainty measures. Algorithm 1 is about computing the roughness and the degree of dependence of single granularity NDRS and NRS in IFODIS. Firstly, an intuitionistic fuzzy ordered decision information system (IFODIS) $\widetilde{I}^{\geq}=(U,AT\cup\{d\},F)$ is inputted as a testing system, where the granulation $A_i\subseteq AT(i=1,2,\ldots,n)$. Then in step 2, we compute all decision classes $U/d=\{D_1,D_2,\ldots,D_m\}$ and the target set X that meets the conditions of the problem. From steps 4–6, we calculate neighborhood dominance class and neighborhood class for every x. The steps 7–10 are to initialize the lower and upper approximations as $\emptyset$. The steps 11–17 are compute lower approximations and upper approximations of NDRS and NRS according to Definition 3.1. In the steps 18-22, we first initialize the degree of dependence as $\emptyset$, and then we obtain the roughness and the dependence degree of NDRS and NRS from Definitions 3.2 and 3.3. Finally, steps 23–26 are to return the roughness and the dependence degree in IFODIS.

Based on the Algorithm 1, the lower and upper approximations of every decision class can be obtained. According to Proposition 5.1, and the relationship between NDRS and MNDRS, we design the Algorithm 2 to compute the roughness and the dependence degree of OMNDRS and PMNDRS. Firstly, we input an IFODIS $\widetilde{I}^{\geq}=(U,AT\cup\{d\},F)$ as a testing system, the target set $X=\widetilde{[x^*]_{A_i}^{\delta\geq}}-\{x^*\}$, and the decision class $U/d=\{D_1,D_2,\ldots,D_m\}$. Then we can obtain the lower and upper approximation sets by Algorithm 1 considering granularity $A_i(i=1,2,\ldots,n)$, and discuss two kinds of MNDRS. In the steps 1–4, we initialize the lower and upper approximation sets of OMNDRS and PMNDRS. Steps 5–11 are to compute the lower and upper approximation sets of OMNDRS and PMNDRS. In steps 12–15, the roughness of OMNDRS and PMNDRS can be acquired by Definition 5.1. From steps 16 to 17, the dependence degree of OMNDRS and PMNDRS are initialized as $\emptyset$. In steps 18-21, the dependence degree of OMNDRS and PMNDRS are calculated by Definition 5.2. Finally, the roughness and the dependence degree of OMNDRS and PMNDRS are returned in steps 22-24.

According to the time complexity of Algorithm 1 and 2 in Table 4, we can know the time complexity of NRS and NDRS maintains the same level. The computation of MNDRS is based on Algorithm 1, hence, the total time complexity of MNDRS is more than that of NRS and NDRS.

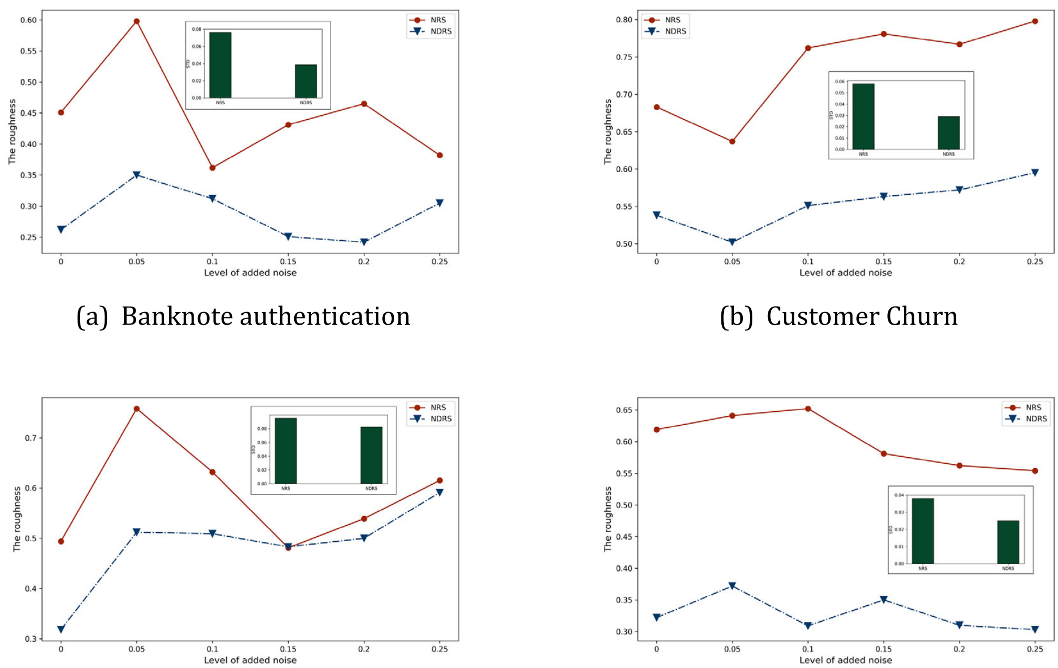
7 Experiments and Analysis

In this section, We will perform a series of experiments through the designed algorithms to verify the effectiveness and the applicability of the proposed methods. we download nine datasets from UCI (http://archive.ics.uci.edu/ml/datasets.php) database. To analyze the robustness of the proposed method, we randomly select four datasets to add noise data proportionally in our experiments. Besides, the superiority of the NDRS and MNDRS will be further illustrated using these nine UCI datasets. The details of datasets are outlined in Table 5. This experimental results are implemented on a personal computer with processor (2.5 GHz Intel Core i5) and memory (8GB 2133MHz DDR4). The platform of experiment environment is Python 3.8.

We should not ignore that the background of this research is IFODIS. The intuitionistic fuzzy value is composed of the degree of membership, the degree of non-membership, and the degree of hesitation. So we need to construct the intuitionistic fuzzy data based on the download data sets. To ensure feasibility and rationality, we first normalize the values in the datasets to the value between 0 and 1 as the degree of membership. Since it is acknowledged the hesitation of different objects in reality is different and cannot be determined, we intend to use random

Table 5 The testing data sets

Data sets	Data Source	Objects	Attributes	Decision class
Immunother	UCI	90	7	2
Wine	UCI	178	13	3
Glass	UCI	214	9	6
Haberman	UCI	306	3	2
Banknote authentication	UCI	1372	4	2
Wireless	UCI	2000	7	4
Customer Churn	UCI	3334	10	2
Wine Quality	UCI	4898	11	7
HTRU	UCI	17899	8	2



(a) Banknote authentication (b) Customer Churn

(c) Wine (d) Wine Quality

Fig. 5 The robustness of models with different noise levels

numbers to describe the degree of hesitation. Besides, the value of each object under the decision attribute d in the data sets remains unchanged. Finally, the IFODIS can be obtained.

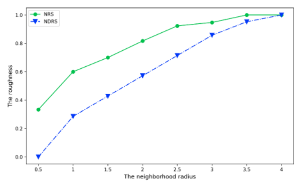
7.1 The Robustness Evaluations of NDRS Model

The noise of these datasets is obtained by increasing the hesitation degree of IFS by 0.05 each time. Subsequently, the roughness metrics of NRS and NDRS are computed for different levels of noise datasets. The experimental results are shown in Fig. 5, with each subplot showing the roughness of the model with different noise levels.

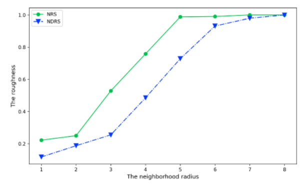
From Fig. 5, we can intuitively observe that the fluctuation of NDRS is smaller than NRS with the increase of noise levels. In addition, we set about to display the standard deviation (STD) of every computation results. Meanwhile, it is obvious to find that STD of NDRS is smaller than NDR. Therefore, it comes to a conclusion that the robustness of NDRS model is better than NRS model.

7.2 The Superiority Evaluations of NDRS Model

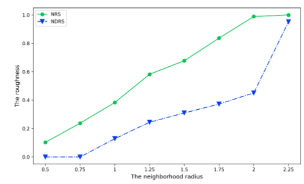
In this subsection, the roughness and dependence of the NDRS and NRS are respectively compared to verify the effectiveness of NDRS model. What cannot be ignored is



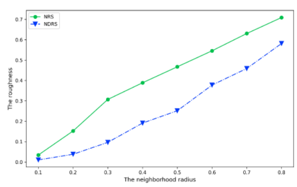
(a) Immunother



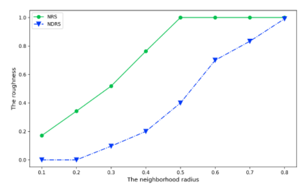
(b) Wine



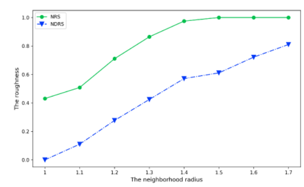
(c) Glass



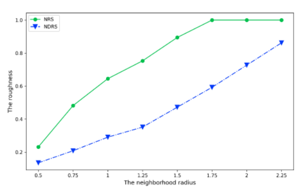
(d) Harberman



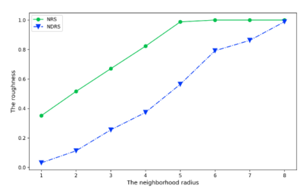
(e) Banknote authentication



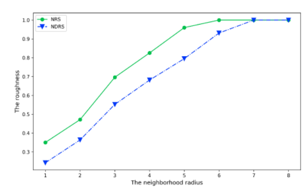
(f) Wireless



(g) Customer Churn



(h) Wine Quality



(i) HTRU

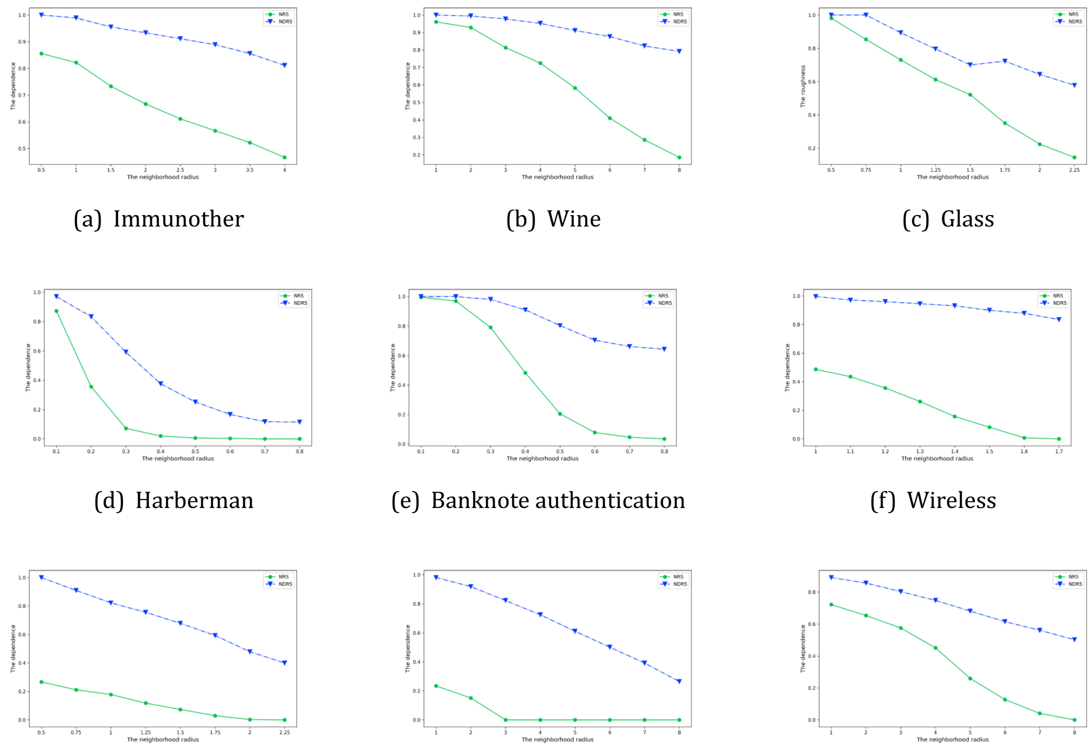
Fig. 6 The roughness of NRS and NDRS at different neighborhood radius

the selection of target concepts and attributes in the process of conceptual approximation. As a result of the different distribution of different datasets, we set the membership degree of the standard object x^* to be greater than or equal to the minimum membership value, and the non-membership degree to be less than or equal to the maximum non-membership value in the intuitionistic fuzzy data, then the target set X under different conditions can be determined as $X = \widetilde{[x^*]}_{A_i}^{\delta\geq} - \{x^*\}$. In order to ensure the reliability of the experiment results, we select $A_1 = AT - \{a_1\}$ in each dataset during the conceptual approximation. Besides, we perform parameter analysis and vary the radius of neighborhood dominance classes and neighborhood classes to obtain a more intuitive and clear experimental comparisons for each dataset. The process of experiments are displayed in Figs. 6 and 7.

In Fig. 6, we can directly find that with the increase of the neighborhood radius, the roughness of NRS and NDRS will also become larger. Meanwhile, the roughness of NRS is always greater than or equal to the roughness of NDRS with the variation of neighborhood radius. As shown in Fig. 7, consistent with the expected conclusion, the dependence of NDRS maintains a higher level compared with NRS, this results show that the NDRS has a better classification accuracy. Consequently, it comes to a conclusion that NDRS is very feasible and effective for sample selection and conceptual approximation tasks (Figs. 8, 9).

7.3 The Comparison Evaluations of NDRS and MNDRS Models

In the datasets, there are $2^{|AT|}$ conditional attribute subsets in each information system $\widetilde{I}^{\geq} = (U, AT \cup \{d\}, F)$ has $|AT|$. For convenience of comparison, we select two conditional attribute subsets as granulations, where two granulations can be expressed as A_1 and A_2. The attribute subsets are $A_1 = AT - \{a_1\}, A_2 = AT - \{a_2, a_3\}$. The experimental process consists of two parts, one is the comparison of the roughness and dependence of the NRS and NDRS, and the other is the comparison of the uncertainty measurement between NDRS and MNDRS. The granulations considered in two kinds of MNDRS models are A_1 and A_2, and the granulation in NDRS and NRS is A_2. The selection of the target set is similar to Sect. 7.2. The computation results of Algorithms 1 and 2 are showed in Table 6.



(a) Immunother (b) Wine (c) Glass

(d) Harberman (e) Banknote authentication (f) Wireless

(g) Customer Churn (h) Wine Quality (i) HTRU

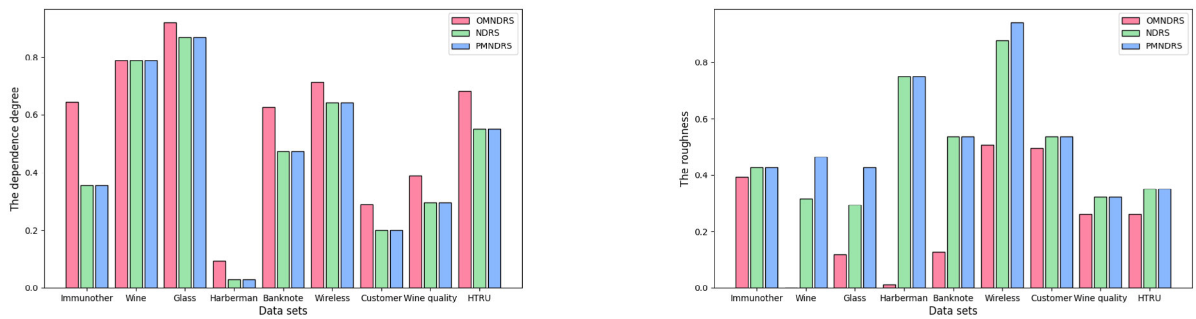
Fig. 7 The dependence of NRS and NDRS at different neighborhood radius

According to the calculation results of the roughness and the dependence degree in Table 6, Figs. 3 and 4 can be drawn. In Fig. 3, It is shown that the dependence degree of three rough set models is decreasing according to the order of OMNDRS, single-granularity NDRS and PMNDRS. As in Proposition 5.4, the degree of dependence of PMNDRS is smallest, the dependence degree of OMNDRS is largest. And the dependence degree of NDRS is between PMNDRS and OMNDRS. Therefore, under the OMNDRS, the percentage of objects classified into the positive region is greater than single-granularity NDRS and PMNDRS.

In Fig. 4, we can find that roughness of OMNDRS is the smallest, the roughness of PMNDRS is the largest. And the roughness of single-granularity NDRS is between OMNDRS and PMNDRS. In the conceptual approximation, the approximate set become increasingly rough in the order of OMNDRS, single-granularity NDRS, and PMNDRS. In practical applications, we should flexibly use single-granularity or multi-granulation rough sets to approximate concepts.

In the selection of the target set, the roughness of the approximate set will change with the neighborhood radius of X. We take the data set "Wine" as an instance to obtain the calculation results under different neighborhood radius of X as shown in Fig. 5. It is easy to find that the roughness of three rough set models is decreasing as the neighborhood radius of X varies, and the roughness of OMNDRS changes more significantly with the increase of the neighborhood radius of X. When the neighborhood radius of X reaches 15, the minimum roughness of OMNDRS can be obtained. The results show the influence of the size of the target set X on the roughness of the three rough set models and verify the importance of target set selection.

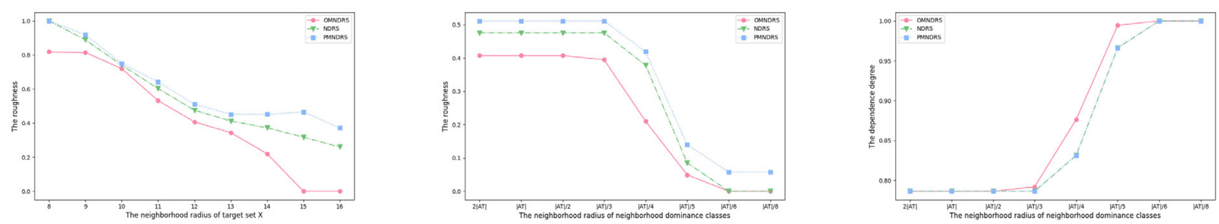
From the calculation results based on "Wine" data set in Figs. 6 and 7, the changes of the roughness and the dependence degree of the three rough set models also depend on the neighborhood radius of the neighborhood dominance classes. We compare the changes of the roughness and the dependence degree of the three rough set models under the neighborhood dominance classes with different neighborhood radius, and we can get that the roughness of three rough set models is decreasing as the neighborhood radius of neighborhood dominance classes ranges from $|AT|$ to $|AT|/8$. When the neighborhood radius of neighborhood dominance classes is $|AT|/3$ or above, the roughness of the three rough set models reaches the maximum. When the neighborhood radius of X is $|AT|/6$ or below, the minimum roughness of the three rough set



(a) The degree of dependence about three rough set models

(b) The roughness of three rough set models

Fig. 8 The evaluation metrics of the proposed models



(a) The changes of roughness with the neighborhood radius of X

(b) The changes of the roughness with the attributes

(c) The changes of the dependence degree with the attributes

Fig. 9 The uncertainty metrics of proposed models in relation to the neighborhood and attributes

Table 6 The computation results of Algorithm 1 and 2

Data sets	Roughness			Dependence degree		
	NDRS	OMNDRS	PMNDRS	NDRS	OMNDRS	PMNDRS
Immunother	0.4270	0.3933	0.4270	0.3556	0.6444	0.3556
Wine	0.3175	0	0.4654	0.7865	0.7865	0.7865
Glass	0.2941	0.1176	0.4286	0.8692	0.9206	0.8692
Haberman	0.7484	0.0131	0.7484	0.0294	0.0948	0.0294
Banknote authentication	0.5372	0.1294	0.5372	0.4745	0.6268	0.4745
Wireless	0.8785	0.5077	0.9423	0.6430	0.7127	0.6430
Customer Churn	0.5381	0.4961	0.5381	0.2010	0.2895	0.2010
Wine quality	0.3224	0.2623	0.3224	0.2973	0.3873	0.2973
HTRU	0.3521	0.2623	0.3521	0.5510	0.6835	0.5510

models can be acquired. In practical problems, we can reduce the roughness of the approximate set to obtain the required approximate set by adjusting the neighborhood radius of the neighborhood dominance classes.

Also, Fig. 7 shows that the degree of dependence of three rough set models is increasing as the neighborhood radius of neighborhood dominance classes decreases. When the neighborhood radius of neighborhood dominance classes is $|AT|/3$ or above, the dependence degree of the three rough set models is almost maintained at a minimum level. When the neighborhood radius of neighborhood dominance classes is $|AT|/6$ or below, the denpendence degree of the three rough set models reaches the maximum. Besides, the result shows that as the neighborhood radius of neighborhood dominance classes becomes smaller, the dependence degree of OMNDRS is always greater than that of PMNDRS and NDRS. Through comparing the changes of the roughness and the dependence degree with different neighborhood radius of neighborhood dominance classes, we can obtain the validity and correctness of our proposition and definition more closely.

8 Conclusion

The classical neighborhood relation constitutes a cover of the universe. It is able to reduce the influence of noisy samples, and select samples that are close to the original object. However, there is a partial order relationship in the actual problems, the classical neighborhood relation cannot solve the problem. Therefore, we introduced the intuitionistic fuzzy neighborhood dominance relation, which combines the classical neighborhood relation with the dominance relation. Based on the intuitionistic fuzzy neighborhood dominance relation, we discussed the single-granularity NDRS model and apply it to actual cases. Then we extended the single-granularity NDRS model to OMNDRS and PMNDRS model. To better illustrate the effectiveness of proposed method, we designed algorithms and experiments to evaluate the model from multiple aspects. Furthermore, we are committed to further improving the applicability of proposed method and continuing to extend the theoretical basis of the model.

In our further research, we will consider applying NDRS to the field of feature selection and introduce dynamic mechanisms, so that the application scope of the model can be extended.

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